Strategic Foundations of Prediction Markets and the Efficient Markets Hypothesis*

Job Market Paper

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Abstract

This paper studies information aggregation in pure common value double auctions with a continuum of traders. This trade environment captures some of the main features of prediction markets. The population includes both strategic traders and non-strategic (naïve) agents whose bidding behavior is not influenced by opponents' equilibrium strategies. Existence and uniqueness of monotone equilibrium prices is shown under mild conditions on the distribution of naïve bids. In any such equilibrium, the mapping from asset values to prices has a domain split into two distinct areas: a revealing region, where prices equal values, and a non-revealing region. There is a strictly positive lower bound on the share of naïve traders below which prices are always fully revealing and an upper bound beyond which prices are almost nowhere revealing. This indicates that, contrary to prevailing views, non-negligible levels of noise or liquidity trade are compatible with perfect information aggregation, although even moderate levels of noise can lead to nowhere revealing prices. An empirical method to distinguish between the revealing and non-revealing regions is suggested.

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1 Introduction

Markets have long been touted not only for their role in allocating goods, but also for their properties as information processors. The *efficient markets hypothesis* (Fama (1970)) postulates that prices in competitive markets "fully reflect" all the available information, which

> "... never exists in concentrated or integrated form, but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all the separate individuals possess."
>
> Hayek (1945, p. 519).

Based on this conjecture, exchange institutions designed with the sole purpose of forecasting future events, commonly referred to as *prediction markets*, have emerged in the last two decades.[1] Although empirical evidence suggests that market prices seem to perform well as information aggregators,[2] the mechanism by which such aggregation takes place is not yet clearly identified. A major reason for this gap is that economics lacks a theory of price formation in markets. Most models either leave the price setting mechanism unspecified (rational expectations equilibrium (REE) models) or assume the existence of a market maker, who sets prices by making inferences on the amount of information individual traders possess (market microstructure models). Existing research on information aggregation in auctions provides an explicit mechanism that links individual trader actions to market prices. However, it has focused primarily on single auctions, which do not account for the two-sided nature of most asset markets.[3]

I study price formation in markets by modeling them as common value double auctions (CVDA) in which risk-neutral traders receive a private signal stochastically related to the value of the security traded.[4] The reasons behind this choice are threefold. First, double auctions provide an explicit mechanism by which individual trade decisions translate into prices. Moreover, this mechanism resembles existing markets, given that both buyers and sellers post offers to respectively buy and sell

[1]Some examples are the Iowa Electronic Markets (IEM) for presidential elections, the Hewlett-Packard internal market to predict future sales and Hollywood exchange, a virtual currency market aimed at forecasting movie ticket sales. I refer the reader to Wolfers and Zitzewitz (2004) for a more comprehensive list of existing markets.

[2]See, for instance, Forsythe, Nelson, Neumann, and Wright (1992), Berg, Forsythe, Nelson, and Rietz (2005), or Berg, Nelson, and Rietz (2003) for evidence from the IEM, and Chen and Plott (2002) for a study of the internal market at Hewlett-Packard.

[3]An important exception is the double auction model of Reny and Perry (2006), which provides theoretical support to the existence of fully revealing REE.

[4]Common values means that the (unknown) value of the asset is the same for all traders, although their expectations do not coincide when they receive distinct signals. In contrast, private values imply that each agent values the asset differently and knows exactly her own valuation.

units of the asset. Second, having a common value component is essential to study information aggregation, if we understand it as the process of aggregating, through market prices, individual pieces of information about the unknown value of the asset.[5] Finally, if the efficient markets hypothesis is correct, pure common values plus risk neutrality imply that security prices in prediction markets can be directly interpreted as estimates of some parameter of the probability distribution of the events to be forecasted.

In order to test how robust information aggregation is to the presence of boundedly rational agents, I introduce heterogeneity in the traders' population by having both strategic traders and naïve traders. These naïve traders can be seen as the analogue of noise in REE models. I provide a very general definition of naïve bidding behavior, which includes the noise or liquidity traders used in market microstructure models as a special case.[6] Thus, their presence allows the comparison of existing results on information aggregation with the predictions of my model. In addition, naïve traders rule out no trade equilibria, which always exist in double auctions in which all traders are strategic.

A known issue in double auctions is that traders' ability to affect prices in finite agent environments makes equilibrium analysis quite intractable. However, since it vanishes as the market grows, I look at a limit case with a continuum of agents.[7,8]

I characterize (increasing) monotone equilibrium prices in this continuum economy and show that they exist and are unique among the class of monotone prices when the distribution of naïve bids satisfies a mild monotonicity condition. Furthermore, I show that in any monotone equilibrium strategic traders place their bids in regions of the bidding space where prices are equal to asset values or outside the range of prices. Accordingly, prices are characterized by having its range partitioned into two distinct regions: a revealing region where prices equal asset values and a

[5] Recent models of prediction markets proposed by Manski (2004), Gjerstad (2005) and Wolfers and Zitzewitz (2006) assume that agents have pure private values (referred to as beliefs). In this setting, each agent knows exactly her valuation of the asset, which differs across agents. Since there is no individual uncertainty, aggregation of beliefs cannot take place and, instead, these analyses look at how close prices are to the mean belief.

[6] The defining feature of naïve traders is that they follow a fixed bidding strategy, regardless of what the other traders do in equilibrium. That is, unlike strategic traders, they do not best respond in equilibrium. Moreover, since the analysis presented below applies to a wide class of naïve bid distributions, I do not require the naïve population to be homogeneous. For instance, the results of my model would hold for a bid distribution that is arbitrarily close to the distribution of bids generated by any mix of level-k agents.

[7] This paper should be regarded as part of a project aimed at analyzing information aggregation in large markets. The study of the limit economy provides insight into the informational content of prices as a function of the degree of noise trade, while subsequent research would explore the convergence of prices in finite markets to prices in the limit economy.

[8] It is important to note that when agents are price takers, the population of naïve traders can include pure private value strategic traders, given that bidding their own valuation is a weakly dominant strategy.

non-revealing area where prices differ from values and are completely determined by naïve bids. There are three distinct scenarios: for small shares of naïve traders, prices are fully revealing (i.e. they equal asset values). When there is a moderate presence of naïve bidders, the equilibrium price function has both a revealing and a non-revealing region. Finally, if the share of naïve traders surpasses some threshold they always determine the price.

This result represents a middle ground between two opposing views about the relationship between liquidity or noise trade and the informational content of prices. According to some models (Kyle (1985, 1989)), the introduction of noise prevents the market from collapsing by precluding prices from fully revealing asset values. On the other hand, REE models predict that, as long as there is a positive mass of risk-neutral traders, prices will be perfectly informative (Grossman (1976), Hellwig (1980)). In the double auction setting, perfectly revealing equilibria are compatible with non-negligible levels of noise trade. However, prices can be quite uninformative for moderate shares of naïve traders, which is at odds with the idea that a small presence of sophisticated traders suffices to get full information aggregation.

It is worth mentioning that, although I restrict my analysis to risk-neutral agents, introducing risk aversion should not alter its main conclusions. This is so because the bidding behavior of strategic traders is not substantively influenced by their attitudes toward risk. Hence, the forecasting properties of prediction markets may not depend on eliciting risk neutrality.

The characterization of equilibrium prices provided here lends itself to the development of an empirical test aimed at assessing how accurately prices reflect asset values. Since strategic traders place bids only in the revealing region while naïve traders generally bid in both regions, a simple method to identify revealing prices can be devised, based solely on data on prices and the density of bids in the neighborhood of prices.

This paper is organized as follows. First I describe a typical prediction market, highlighting its relevant features. I then look at existing theories that address information aggregation through the price mechanism. The common value double auction model is laid out in section two. Section three presents a no trade theorem. Section four provides the characterization of equilibrium prices in a continuum agent economy. An empirical test of information aggregation is then suggested. Before concluding, I briefly discuss convergence issues regarding finite double auctions.

1.1 Morphology of a Prediction Market

Assume our goal is to predict the outcome of a U.S. presidential election, to be held at time T. Primaries are over and there are two candidates left, George and John. We would like to estimate at any time $t < T$ the probability of each candidate winning the popular vote. Denote such probabilities by $\mathbb{P}_t^G$ and $\mathbb{P}_t^J = 1 - \mathbb{P}_t^G$, respectively. Each individual agent has some information about $\mathbb{P}_t^G$, denoted $\mathscr{F}_{i,t}$.

For this purpose, we set up a futures market in which agents can post and accept offers to trade two futures contracts: the "George security" (denoted by G), which pays \$1 at time T if George wins the popular vote and \$0 otherwise; and the "John security" (J), which pays \$1 if John wins and \$0 otherwise. According to the efficient markets hypothesis (henceforth EMH), the equilibrium price of each security at any time t (p_t^j, $j = G, J$) is such that the expected future price conditional on all the existing information $\mathscr{F}_t = \bigcup_i \mathscr{F}_{i,t}$ satisfies

$$[1 + \mathbb{E}(r_{t+1}^j | \mathscr{F}_t)] p_t^j = \mathbb{E}(p_{t+1}^j | \mathscr{F}_t), \tag{1}$$

where r_{t+1}^j is the one period return to security j at time $t+1$, which will depend on agents' (identical) preferences. Under risk neutrality and pure common values, $r_t^j = 0$ for all t and $p_T^j = 1_{\{j \text{ wins the popular vote}\}}$,[9] given that neither security pays dividends but instead has a liquidation value at T of \$1 or \$0, depending on the election outcome. This implies that (1) reduces to

$$p_t^j = \mathbb{E}(p_T^j | \mathscr{F}_t) = \mathbb{E}(1_{\{j \text{ wins the popular vote}\}} | \mathscr{F}_t) =: \hat{\mathbb{P}}_t^j. \tag{2}$$

Therefore, if the efficient market hypothesis is correct and traders are money-maximizers with no insurance motives, the price of each security provides the "best" estimate of the winning probability of its associated candidate, given the information available at that period.

Guided by this conjecture, prediction markets are usually set up as on-line futures markets with specific features so as to induce risk-neutral behavior among participants.[10] Trading rules resemble those of existing stock exchanges, which are in essence continuous-time double auctions: traders can either post offers to sell or buy each of the securities (known as *asks* and *bids*, respectively) or accept outstanding offers. An ask (bid) specifies the maximum number of units of the security to sell (buy) and the minimum (maximum) price to be accepted. All valid asks are ranked from lowest to highest price, whereas all valid bids are ranked from highest to lowest. The two outstanding offers are the ask with the lowest price and the bid with the highest price. If an outstanding ask (bid) is accepted, the next highest ranked ask (bid) becomes outstanding. When an outstanding offer is accepted, the price of the ensuing transaction is the price specified in this offer. At each moment, traders are informed of the outstanding offers and the price of the last transaction. Additional information such as trade volume, price and bid/ask spread histories is also available.

The empirical evidence seems to suggest that prices perform well as forecasts. Regardless of the specific characteristics of the prediction market (size, virtual vs.

[9] $1_{\{.\}}$ is the indicator function.

[10] In some of these markets there are tight limits on the amount of money that can be invested. For instance, the investment limit at the IEM is \$500. In addition, transaction costs in these markets are negligible.

real currency, type of event to be forecasted), the short and long-run forecasting properties of prices appear to be better than existing benchmarks, mainly experts' forecasts and opinion polls. For instance, Forsythe, Nelson, Neumann, and Wright (1992), Berg, Forsythe, Nelson, and Rietz (2005) and Berg, Nelson, and Rietz (2003) find that market prices were consistently closer than opinion polls to actual vote shares at the IEM election markets. With regards to experts' forecasts, Chen and Plott (2002) show that price forecasts in the (small) internal market set up by Hewlett-Packard were closer to actual sales of the company than the official company forecast.[11] Finally, experimental evidence in oral common value double auctions suggests that prices aggregate information, thus backing the EMH (Plott and Sunder (1988), Forsythe and Lundholm (1990), Guarnaschelli, Kwasnica, and Plott (2003)).

1.2 Theoretical Foundations of the EMH

Despite these empirical findings, REE models, on which the EMH is based, do not describe price formation as a function of traders' actions. Therefore, how information may be reflected in market prices is left unexplained.[12]

There are two strands of economic theory that analyze price informativeness by linking prices to individual trading behavior: market microstructure models and auction theory. In market microstructure models such as the canonical models of Glosten and Milgrom (1985) and Kyle (1985) prices typically arise as a result of the interaction between competitive market makers and individual traders. In this setting, uninformed market makers set prices according to a zero profit rule by making inferences about the information traders may have.[13] Two salient features are the sequential nature of these models and the heterogeneity of the trader population. The latter usually consists of informed, strategic traders and uninformed traders, who can be strategic or not (noise traders) depending on the model. This literature captures important aspects of the price formation process in some capital markets in which specialists operate the market. In addition, they provide insights into how traders' information gets reflected into prices and the timing of this process. Specifically, they show that information is incorporated into prices gradually, allowing informed traders to profit from their privileged information, contrary to the EMH. However, the presence of a non-trivial market maker renders these models unsuited for the analysis of two-sided decentralized institutions in which no specialists are

[11] See Sunstein (2004) and Wolfers and Zitzewitz (2004) for a more general discussion about the empirical evidence regarding prediction markets.

[12] In REE models such as those of Hellwig (1980), and Grossman (1976, 1978), traders observe prices before choosing their demands. Thus, the mechanism by which traders' actions translate into prices is left out of the model.

[13] There is an extensive literature emerging from these two models. Important examples are Easley and O'Hara (1992) and the analysis of Back and Baruch (2004), which shows that Glosten and Milgrom (1985) and Kyle (1985) merge into the same model under some conditions.

present, where all the strategic interaction takes place between individual traders operating without the constraint of a zero-profit rule.

On the other hand, auction theory looks at information aggregation by modeling markets as static common value auctions, which have a very simple institutional structure and pricing rules are predetermined before the auction. Although the static nature of these trade environments precludes the study of information aggregation dynamics, this approach has the advantage of providing an explicit mechanism that links market prices to individual bids. Most research has focused on one-sided common value auctions, starting with the first price auction of Wilson (1977). The main finding is that equilibrium prices converge to the true value of the asset as the number of bidders gets large as long as either the upper bound of the asset value support grows (Wilson (1977) and Milgrom (1979, 1981)) or as the units at auction increase (this is the *double largeness* condition in Pesendorfer and Swinkels (1997)).[14] Information aggregation is caused by agents' inferences about prices based on their private information and on the equilibrium behavior of the other agents. These inferences influence bidding behavior which, in turn, determines prices.

The main drawback of using one-sided auctions to analyze information aggregation in markets is that there is an implicit non-strategic market maker (the seller) in charge of the supply. Double auctions models solve this issue by having both strategic buyers and sellers. However, common value double auctions have proven quite intractable and very little research exists in this area. A remarkable exception is the paper by Reny and Perry (2006), who study the existence of fully revealing equilibrium prices in large mixed value double auctions. They show the existence of approximately fully revealing prices in large double auctions when agents' utility is strictly increasing in the signals agents privately receive.[15] Since the private value component is non-negligible prices do not converge to values as the market grows.

2 The model

There is a continuum of agents, denoted by $\mathcal{T}$. A fraction $\gamma \in (0,1)$ of them are sellers, each of them owning one unit of a security, with the remaining fraction being buyers, willing to buy at most one unit. The value of the security $V \in [0,1]$ is unknown with probability distribution $G(V)$. Each agent receives a private signal $S \in [0,1]$ stochastically related to V. Signals are independent and identically distributed conditional on v, with probability distribution $F(S|v)$.[16]

[14] Kremer (2002) summarizes existing results and extends them to the English auction, while Hong and Shum (2004) study the rates of convergence under both scenarios.

[15] They prove this result for double auctions with finite bid grids.

[16] In what follows, I use uppercase letters to denote random variables (V, S) or cumulative distribution functions (G, F) and lowercase to denote realizations of random variables (v, s_i) or density functions (g, f). In addition boldface letters (e.g. $\mathbf{s}, \mathbf{S}$) denote vectors.

Assumption 1 $G(.)$ *has a* C^1 *density* $g(.)$ *bounded away from 0 in* $[0,1]$*.* $F(\,.\,|.)$ *has a* C^1 *density* $f(\,.\,|.)$ *bounded away from 0 in* $[0,1]^2$*.*

Assumption 2 $f(\,.\,|.)$ *satisfies the strict monotone likelihood ratio property.*

The first assumption implies that the distribution of signals has full support for all values of the asset. That is, a trader receiving a signal $s \in [0,1]$ cannot rule out any asset value in $[0,1]$. The second assumption means that higher signals are more likely than lower signals when the asset value is high.

Buyers and sellers simultaneously submit bids and asks to buy and sell specifying, respectively, the maximum price willing to pay and the minimum price willing to accept. Bids are restricted to be in $[0,1]$.[17] The price p is given by the $(1-\gamma)$-th percentile of the bid distribution. Buyers with bids above p and sellers with asks below p get to trade.[18] If there is a positive mass of bids at p there is the possibility of rationing, i.e. some traders bidding exactly p may not trade. In this case, the traders bidding p who end up with the object are chosen randomly.[19]

A fraction $\eta \in [0,1]$ of both buyers and sellers are naïve traders who do not best respond in equilibrium but rather use a fixed bidding rule. The remaining mass of agents are risk-neutral, strategic traders, i.e. they best respond in equilibrium. The bidding behavior of naïve traders is summarized by the probability distribution of their bids, $H(.|v)$. I assume that $H(.|v)$ is continuous, weakly monotonic with respect to asset values and has the same connected support for all v.

Assumption 3 $H(.|v)$ *has full support in* $[\underline{b}^H, \overline{b}^H] \subseteq [0,1]$ *for all* $v \in [0,1]$*, with* $\underline{b}^H < \overline{b}^H$*.* $H(.|.)$ *is* C^1 *in* $(\underline{b}^H, \overline{b}^H) \times [0,1]$ *and absolutely continuous in* $[0,1]^2$*.*

This assumption implies that the distribution of naïve bids is atomless. The full support assumption implies that $H(.|v)$ is strictly increasing in $(\underline{b}^H, \overline{b}^H)$ for all $v \in [0,1]$, i.e. there are no intervals between the lowest and highest naïve bids where the mass of bids is zero.

Assumption 4 $H(b|.)$ *is non-increasing in* $[0,1]$ *for all* $b \in [0,1]$*.*

Assumption 4 implies that, for all v, v' such that $v > v'$, $H(.|v)$ first order stochastically dominates $H(.|v')$. It means that naïve traders tend to bid higher when the value of the asset is higher.

Examples of naïve bidding satisfying the above assumptions include the typical random or noise traders commonly used in finance models, who bid according to

[17] This assumption is without loss of generality since bids outside the unit interval are weakly dominated by bidding either zero or one.

[18] In the remainder of the paper I use the term "bid" to refer both to seller asks and buyer bids.

[19] Reny and Perry (2006) use the same tie-breaking rule.

the bidding rule $\beta^n(s) \sim U[0,1]\ \forall s$,[20] and traders bidding according to their interim private beliefs, $\beta^n(s) = \mathbb{E}(V|s)$. The latter are similar to traders in the prediction market models of Manski (2004), Gjerstad (2005) and Wolfers and Zitzewitz (2006). In those models, traders fail to consider the common value nature of the asset. Accordingly, one could interpret their behavior as strategic, in the sense that, if they deem their interim beliefs as their true valuation of the asset, they would be best responding by bidding $\mathbb{E}(V|s)$.[21] Thus, a double auction in which the trader population is divided into pure private value and pure common value strategic agents constitutes a special case of the double auction environment presented above. Also, the above definition of naïve traders can include any continuous approximation of the distribution of bids generated by a population consisting of any mix of level-k agents (see Crawford and Iriberri (2006) for a definition of level-k thinking in auctions).[22]

Given a profile of bidding (pure) strategies $\beta : [0,1] \times \mathcal{T} \to [0,1]$ with $\beta(s,t)$ denoting the bid of strategic trader $t \in \mathcal{T}$ when she receives signal s, let $B(.|V,\eta)$ be the cumulative distribution function of bids when the share of naïve traders is η and $B\_(p|V,\eta)$ the mass of bids strictly less than p. Accordingly,

$$B(p|v,\eta) := \eta H(p|v) + (1-\eta)\int_{\mathcal{T}}\int_0^1 1_{\{\beta(s,t)\leq p\}} f(s|v)\,ds\,d\mu, \tag{3}$$

and

$$B\_(p|v,\eta) := \eta H(p|v) + (1-\eta)\int_{\mathcal{T}}\int_0^1 1_{\{\beta(s,t)< p\}} f(s|v)\,ds\,d\mu, \tag{4}$$

where μ is a suitable (atomless) measure on $\mathcal{T}$.

The asset value V determines the distribution of signals $F(.|V)$. Given that there is a continuum of traders receiving i.i.d. signals, by the law of large numbers, the profile of signals received by traders coincides with the whole distribution of signals conditional on V.[23] Accordingly, given strategy profile $\beta(.,.)$, the market clearing price is completely determined by V. Hence, for all $v \in [0,1]$ the market price is

[20] Their bid distribution is $H(b|v) = b$ for all v, which weakly satisfies *Assumption 4*.

[21] This is true given that in a continuum economy agents are price takers. Thus, it is optimal for a buyer (seller) to bid her private value in order to maximize her gains from trade.

[22] The distribution of bids generated by such population can include atoms and therefore violate *Assumption 3*. However, such distribution can be approximated by an atomless distribution satisfying the above assumptions.

[23] As pointed out by Judd (1985) there are measurability problems when dealing with a continuum of random variables. While acknowledging those issues, I do not address them in the analysis presented here. Hammond and Sun (2006) propose extending the usual product probability space to one that retains the Fubini property so that measurability is restored. For instance, as shown by Sun (1996), we obtain the exact law of large numbers if we assume that $\mathcal{T}$ is a hyperfinite set (thus having cardinality continuum) and extend the standard product measure space defined on $[0,1] \times \mathcal{T}$ to the corresponding nonstandard Loeb space.

given by the function $\rho : [0,1]^2 \to [0,1]$ that satisfies

$$(1-\gamma) \in [B_{\_}(\rho(v,\eta)|v,\eta), B(\rho(v,\eta)|v,\eta)]. \tag{5}$$

The payoff functions for a (strategic) buyer t and a seller t' are, respectively,

$$\begin{aligned} \pi^{buy}(s,t) &:= \mathbb{E}((V-\rho(V,\eta))1_{\{\beta(s,t)>\rho(V,\eta)\}}|s) \\ &+ \mathbb{E}((V-\beta(s,t))\lambda(\beta(s,t),V)1_{\{\beta(s,t)=\rho(V,\eta)\}}|s), \end{aligned} \tag{6}$$

and

$$\begin{aligned} \pi^{sell}(s,t') &:= \mathbb{E}((\rho(V,\eta)-V)1_{\{\beta(s,t')<\rho(V,\eta)\}}|s) \\ &+ \mathbb{E}((\beta(s,t')-V)(1-\lambda(\beta(s,t'),V))1_{\{\beta(s,t')=\rho(V,\eta)\}}|s), \end{aligned} \tag{7}$$

where $\lambda(b,v)$ represents the probability of getting the object given bid b and asset value v when $\rho(v,\eta)=b$.

3 Equilibrium Prices

In this section I investigate how well equilibrium prices forecast asset values as a function of the presence of naïve traders (η) and of their specific bidding behavior ($H(.|.)$). Accordingly, I restrict my attention to equilibria with prices $\rho(v,\eta)$ that are increasing in v (henceforth monotone equilibria). The two main results are stated in *Propositions 1* and *2*. The first provides a characterization of monotone equilibrium prices, whereas the second shows existence and uniqueness of such prices. All proofs are relegated to the *Appendix*.

The characterization of equilibrium prices provided below is driven by the inability of a single trader to affect prices when there is a continuum of agents. Price taking behavior induces two key features of payoff functions (6)-(7): (i) buyers and sellers have the same preference ranking over bids (*Lemma 1*), and (ii) bidding behavior is oriented to maximize the probability of trading in favorable conditions while avoiding undesired trades, considering prices fixed. The latter, coupled with increasing prices, leads strategic traders to avoid bidding in areas of the price range where prices are not equal to asset values, i.e. where prices are non-revealing.

Lemma 1 (Symmetric Preferences) *Buyers and sellers receiving the same signal $s \in [0,1]$ have the same ranking over bids in $[0,1]$.*

To get some intuition on both the symmetry of preferences and the incentive to avoid bidding in non-revealing regions, consider the price function depicted in *Figure 1*. The range of prices consists of a revealing interval $[0,p_1]$ and a non-revealing interval $(p_1,p_2]$. Prices are greater than values whenever $\rho(v,\eta)$ is above the diagonal

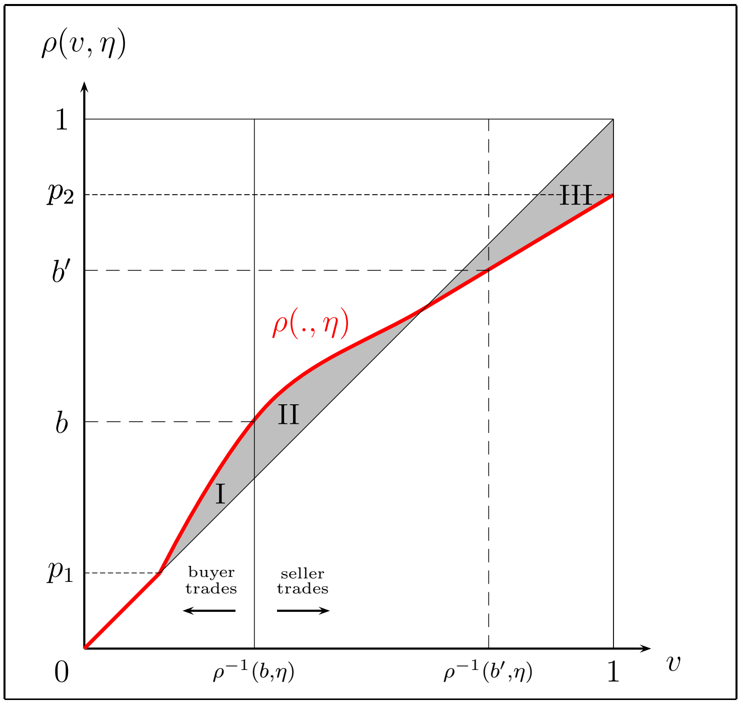

Figure 1: Strategic Bidding

and viceversa. Assuming there is no rationing, the payoff for a buyer bidding b when she receives signal s is the expected value of the difference $v-\rho(v,\eta)$, conditional on s, for all prices below b. That is, it is the expected value of shaded area I. For a seller with signal s and bid b, the payoff is obtained by integrating the difference $\rho(v,\eta)-v$ for prices above b, i.e. the expected difference between transactions involving prices above values (area II) and transactions for which $\rho(v,\eta) < v$ (area III).

The rationale behind symmetric preferences is the following: for a seller with signal s, the payoff for trading the object when prices fall within two alternative bids b and b' is the negative of the payoff for a buyer receiving the same signal. In addition, when a seller bidding b trades (because $\rho(v,\eta) > b$) a buyer bidding b does not trade. If a seller with signal s strictly prefers bid b to bid $b' > b$ it is because the expected payoff, conditional on s, of trading at prices between b and b' is positive.[24] But then, a buyer with the same signal would rather avoid trading at those prices by also bidding b.

Consider now a seller who places a bid in a non-revealing area, e.g. by bidding b. She has an incentive to deviate and bid either in $[0,p_1]$ since all transactions in area I involve $\rho(v,\eta) > v$, or to bid in $[p_2,1]$ if, conditional on her signal, the expected difference between the gains in areas I and II and the losses in area III is negative. In

[24]That is, for values between $\rho^{-1}(b,\eta)$ and $\rho^{-1}(b',\eta)$, where ρ^{-1} denotes the inverse image of the price function.

the latter case, by bidding above p_2 she abstains from trading and gets zero payoffs. On the other hand, if a seller bids b' she is engaging in negative transactions, which can be avoided by bidding above the area where $\rho(v,\eta) < v$ (i.e. in $[p_2, 1]$) or can be compensated with gains from areas I and II (by bidding in $[0, p_1]$). By the symmetry of preferences, no buyer would bid b or b'. This also indicates that no strategic trader would bid below a non-revealing interval that starts with $\rho(v,\eta) < v$.

Proposition 1 and *Corollary 1* are a direct consequence of these two key characteristics of traders' payoffs. Let $H(v) = H(v|v)$.

Proposition 1 (Equilibrium Prices) *In any monotone equilibrium of a CVDA satisfying Assumptions 1-4, there is a set* $\mathcal{V} = \bigcup_k [\underline{v}_k, \overline{v}_k]$ *with* $\overline{v}_k < \underline{v}_{k+1}$ *for all* $k = 1, ..., K \leq \infty$ *and a collection of signals* $\{s_k^*\}$ *with* $s_k^* < s_{k+1}^*$ *such that prices are given by*

$$\rho(v,\eta) = \begin{cases} v & \text{if } v \in [0,1] \smallsetminus \mathcal{V} \\ p \text{ s.t. } H(p|v) = \frac{1-\gamma-(1-\eta)F(s_k^*|v)}{\eta} & \text{if } v \in [\underline{v}_k, \overline{v}_k], \end{cases} \tag{8}$$

where all $\underline{v}_k, \overline{v}_k \in (0,1)$ *and* $s_k^* \in [0,1]$ *satisfy:*

$$1-\gamma = \eta H(\underline{v}_k) + (1-\eta)F(s_k^*|\underline{v}_k), \tag{9}$$

$$1-\gamma = \eta H(\overline{v}_k) + (1-\eta)F(s_k^*|\overline{v}_k), \tag{10}$$

and, for all $s \leq s_k^*$ $(s \geq s_k^*)$*,*

$$\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s) \leq 0 \;\; (\geq 0). \tag{11}$$

This result essentially describes monotone equilibria as the succession of non-revealing intervals ($[\underline{v}_k, \overline{v}_k]$), where prices are determined by the naïve bids, and revealing intervals ($[\overline{v}_k, \underline{v}_{k+1}]$) in which all strategic bids within the price range are concentrated. It also establishes that the allocation of strategic bids across the latter intervals is block-monotonic, i.e. lower signal traders bid in lower intervals. Specifically, traders with signals in (s_k^*, s_{k+1}^*) bid in $[\overline{v}_k, \underline{v}_{k+1}]$.[25] It is important to point out that prices may be fully revealing, i.e. $\mathcal{V}$ is the empty set, or completely determined by the distribution of naïve bids, i.e. $\mathcal{V} = [0,1]$. The next corollary further requires that, in any non-revealing interval, prices are above values in the lower portion of the interval and below values in the upper part.

[25] When $\underline{v}_1 > 0$ traders with signals below s_1^* bid in $[0, \underline{v}_1]$ and in $[0, \rho^{-1}(0,\eta)]$ if $\underline{v}_1 = 0$, i.e. outside the price range. Similarly, traders with signals above s_K^* either bid in $[\overline{v}_K, 1]$ (when $\overline{v}_K < 1$) or in $[\rho^{-1}(1,\eta), 1]$ (when $\overline{v}_K = 1$).

Corollary 1 *In any monotone equilibrium with $\mathcal{V}$ non-empty, given $(\underline{v}_k, \overline{v}_k)$, either $\rho(v,\eta) = v$ a.e. in $(\underline{v}_k, \overline{v}_k)$ or there exist v'_k, v''_k with $\underline{v}_k < v'_k \leq v''_k < \overline{v}_k$ such that $\rho(v,\eta) \geq v$ a.e. in $[\underline{v}_k, v'_k]$ with strict inequality in a non-null subset, and $\rho(v,\eta) \leq v$ a.e. in $[v''_k, \overline{v}_k]$ with strict inequality in a non-null subset. Moreover, if this is true for intervals $(\underline{v}_k, \hat{v}]$ and $(\hat{v}, \overline{v}_k)$, then $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in(\hat{v},\overline{v}_k)\}}|s^*_k) \geq 0$.*

Corollary 1 states that prices in intervals where no strategic bids are placed need to begin with prices above values and end with prices below values. When faced with the prospect of bidding just below an interval where values are always above prices, a seller would rather deviate and bid just above that region to avoid trading at those prices. A symmetric reasoning applies when the seller bids just above an interval in which $\rho(v,\eta) > v$. It also says that, if a non-revealing interval consists of two or more disjoint subintervals, each of them beginning with prices above values and ending with prices below values, no strategic bidder bidding below such interval has an incentive to deviate and bid in between two of those subintervals.

Examples of prices that can and cannot be an equilibrium are shown in *Figure 2*.[26]

The proofs of *Proposition 1* and *Corollary 1* hinge upon a series of lemmas in the *Appendix*, which formalize the intuition about strategic bidding presented above. In addition to having symmetric preferences (*Lemma 1*), I show that no strategic bidder would place bids just below a non-revealing interval that starts with prices below values (*Lemma 2*). *Lemma 3* states that strategic bidders avoid placing bids in non-revealing intervals. Finally, as assumed above, no rationing takes place in equilibrium, since any atom is solely created by naïve traders and can only happen in very special cases (*Lemma 4*). In addition, the block-monotonicity of the distribution of strategic bids is a direct consequence of the MLRP property and of *Lemma 2*: if bidding just below a non-revealing interval is profitable for a trader with signal s, it is also profitable for all traders with signals below s.

The next result states that monotone equilibria exist in any CVDA with a continuum of agents satisfying *Assumptions 1-4*. Furthermore, monotone equilibrium prices are unique. Finally, it sheds light on how the presence of naïve traders affect the informational content of prices: there is a strictly positive lower bound on the share of naïve bidders below which prices are fully revealing and there is an upper bound above which prices are always set by naïve bidders and strategic bidders bid outside the price range.

[26] To correctly interpret these figures assume that strategic bids are placed in the intervals where $\rho(v,\eta) = v$.

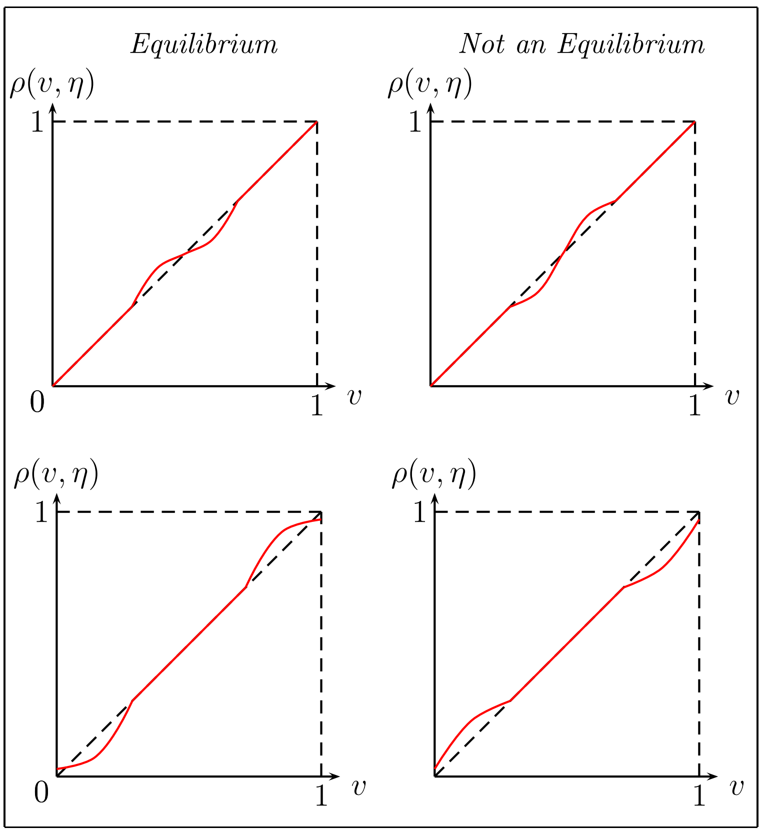

Figure 2: Candidates for Equilibrium Prices

Proposition 2 (Existence of Monotone Equilibria) *Let Assumptions 1-4 be satisfied. Then a monotone Bayesian Nash equilibrium in pure strategies exists for all* $\eta \in [0,1]$ *and the resulting price function* $\rho(.,\eta)$ *is the unique monotone price function that can be supported in equilibrium. Furthermore, there exists* $\underline{\eta} \in (0, \min\{\gamma, 1-\gamma\})$ *such that* $\mathcal{V}$ *is the empty set (fully revealing prices) for all* $\eta < \underline{\eta}$*, and there exists* $\overline{\eta} \leq 1$ *such that* $\mathcal{V} = [0,1]$ *for all* $\eta > \overline{\eta}$ *(*$\overline{\eta} < 1$ *if* $H'(v) \geq 0$ *whenever* $H(v) = 1-\gamma$*).*

Existence of equilibrium is given by the continuity of distributions H, F and expectations, and by the monotonicity of H, F with respect to v. The former guarantees the existence, for each η, of a block-monotonic distribution of strategic bids satisfying the equilibrium conditions of *Proposition 1*. The latter leads to increasing prices when the distribution of strategic bids is block-monotonic. Uniqueness is based on the fact that, due to the strict MLRP, each triplet $(s_k^*, \underline{v}_k, \overline{v}_k)$ satisfying (9)-(11) and *Corollary 1* is unique. An algorithm to obtain the collection $\{(s_k^*, \underline{v}_k, \overline{v}_k)\}_{k=1}^K$ that characterizes equilibrium price $\rho(v,\eta)$ is provided in the proof of *Proposition 2*.

The last part of *Proposition 2* establishes the existence of three types of equilibrium prices depending on the proportion of naïve traders: fully revealing, partially revealing and nowhere revealing prices. To provide some intuition on this result, let the quantile function $\alpha(v,\eta)$ represent the highest signal corresponding to bids at or below v such that $\rho(v,\eta) = v$, assuming block-monotonicity. That is, for all $v \in [0,1]$ such that $\frac{1-\gamma-\eta H(v)}{1-\eta} \in [0,1]$, $\alpha(v,\eta)$ is given by

$$F(\alpha(v,\eta)|v) = \frac{1-\gamma-\eta H(v)}{1-\eta}. \tag{12}$$

For most distributions of naïve bids, $\alpha(.,.)$ has three distinct regions, depending on the value of η (see *Lemma 5* in the *Appendix*).[27] For $\eta \in [0, \underline{\eta}]$, it is increasing with respect to v in $[0,1]$. It is non-monotonic (whenever it is well-defined) in v for $\eta \in (\underline{\eta}, \overline{\eta})$. Finally, it is decreasing for all $\eta \geq \underline{\eta}$. I show that, when $\alpha(.,\eta)$ is increasing everywhere, prices must be fully revealing. This is true because the proportion of naïve traders is too small to create non-revealing intervals starting with prices above values, as required by *Corollary 1*, when the distribution of strategic bids is block-monotonic. On the other hand, prices cannot be revealing in intervals of values where $\alpha(.,\eta)$ is not well-defined or decreasing (*Lemma 6*). For prices to be revealing in some interval $[v_1, v_2]$ in which $\alpha(.,\eta)$ is decreasing, it would be necessary to have a mass of bids below v_2 that is strictly smaller than the mass of bids below $v_1 < v_2$. Moreover, that reduction of mass needs to be greater than

[27] Distributions $H(.|.)$ such that $H'(v) < 0$ for some v are typically multi-modal distributions, with most of the mass concentrated in a small subset of the support. These can be generated, for instance, by almost-jump bidding functions, which imply bidding in a small neighborhood of a finite set of bids for most signals.

$F(\alpha(v_1,\eta)|v_1) - F(\alpha(v_1,\eta)|v_2)$, given that $\alpha(v_2,\eta) < \alpha(v_1,\eta)$. However, this is not possible under the strict MLRP, because the highest possible reduction of mass is obtained by having bid distributions equal to $F(\alpha(.,\eta)|.)$.

Before presenting an example that illustrates how information aggregation can be quite sensitive to the presence of naïve traders, it is worth mentioning two aspects about trade activity in monotone equilibria. Block-monotonicity means that the strategic traders more active in the market are the low signal sellers and the high signal buyers. Since the former tend to bid relatively low and the latter bid relatively high, they engage in trade more often, compared to high signal sellers and low signal buyers. In addition, the density of bids will generally be higher around fully revealing prices, given that strategic traders avoid bidding in non-revealing regions. As suggested in *Section 4*, this could be the basis to develop an empirical test of information aggregation.

I now provide an example to illustrate how the informational content of prices varies with the share and bidding behavior of naïve traders and to provide some intuition for *Proposition 2*.

Example 1 *Consider a CVDA with the following characteristics. V is distributed uniformly in $[0,1]$; the conditional distribution of signals is Beta$(1+v,1)$ (i.e. $F(s|v) = s^{1+v}$);*[28] *each naïve trader bids according to $\beta^n(s) := \frac{3}{5}s^{1/5}$, which is a rough approximation of bidding $\mathbb{E}(V|s)$.*[29]

Given $\beta^n(.)$, the distribution of naïve bids is given by

$$H(p|v) = \begin{cases} \left(\frac{5}{3}p\right)^{5(1+v)} & \text{if } v \leq \frac{3}{5} \\ 1 & \text{if } v > \frac{3}{5} \end{cases} \tag{13}$$

By *Proposition 2*, there exist cutoff points $\underline{\eta}$, $\overline{\eta}$ that determine whether prices would be fully, partially or non-revealing as a function of η. Since $H'(v) \geq 0$ for all v, $\overline{\eta}$ is strictly less than one.

The first thing to note is that, given η, a necessary condition for a partially revealing equilibrium with $\mathcal{V} = [\underline{v}_1, \overline{v}_1]$ is that there exist a signal s_1^* satisfying (9) at three distinct values, namely $\underline{v}_1$, $\overline{v}_1$ and $v_1' \in (\underline{v}_1, \overline{v}_1)$, the latter being the point at which $\rho(v,\eta)$ goes from being above to go below v. Therefore, the function $\alpha(v,\eta)$ given by

$$\alpha(v,\eta) = F^{-1}\left(\tfrac{1-\gamma-\eta H(v)}{1-\eta}|v\right) = \left[\frac{1-\gamma-\eta\left(1_{\{v>3/5\}} + 1_{\{v\leq 3/5\}}\left(\frac{5}{3}v\right)^{5(1+v)}\right)}{1-\eta}\right]^{\frac{1}{1+v}}$$

[28] This distribution satisfies all assumptions except the full support, provided it has positive density in $(0,1)$ rather than in $[0,1]$.

[29] This approximation makes equilibrium computations more tractable without changing any substantive aspect of the analysis.

needs to be three-to-one in some subset of its range. If it is strictly increasing in $[0,1]$, then equilibrium prices will necessarily be fully revealing. Otherwise, if the mass of bids in some interval $[v_1, v_2]$ is reallocated to $[0,1] \setminus [v_1, v_2]$ prices would no longer be fully revealing in the latter set, and a positive mass of strategic bidders would rather deviate. On the other hand, if for some η there exists a signal s such that $\mathbb{E}(v - \rho(v,\eta)|s) = 0$ where $\rho(v,\eta)$ given by $1-\gamma = \eta H(p|v) + (1-\eta)F(s|v)$ for all $v \in [0,1)$ satisfies $\rho(0,\eta) > 0$ and $\rho(1,\eta) < 1$, then $[v_1, v_2] = [0,1]$ fulfils *Corollary 1*, and strategic bids will be confined to $[0,1] \setminus (\rho(0,\eta), \rho(1,\eta))$.[30]

In a symmetric market ($\gamma = 0.5$), I find that $\underline{\eta} \approx 0.016$ and $\overline{\eta} \approx 0.214$. This shows that the range of η compatible with fully informative prices can be quite small. As an illustration, *Figure 3* shows equilibrium prices when 10% of traders are naïve.

The first thing to note is that even with such a low proportion of naïve traders, prices can be substantially different from values in about half of the domain of values. Roughly speaking, the probability that prices reflect the true asset value is about one half in this example.

The mass of strategic bids is split between those with signals below s_1^*, who place bids in $[0, \underline{v}_1]$ and traders with signals above s_1^*, who bid in $[\overline{v}_1, 1]$. This and the fact that buyers and sellers have the same preference ranking over bids implies that sellers with signal $s < s_1^*$ are "active" in the market (i.e. their probability of trading is positive) and get positive expected payoffs while buyers with the same signal are indifferent between trading or not provided they only trade when $\rho(v,\eta) = v$.

The graph of $\alpha(v, 0.1)$ (middle graph of *Figure 3*) provides some intuition on the existence and uniqueness of prices. As mentioned above, $(s_1^*, \underline{v}_1, \overline{v}_1)$ are given by (9)-(11), i.e. $\rho(\underline{v}_1, 0.1) = \underline{v}_1$, $\rho(\overline{v}_1, 0.1) = \overline{v}_1$ and $\mathbb{E}((V - \rho(V,\eta)1_{\{v \in [\underline{v}_1, \overline{v}_1]\}}|s_1^*) = 0$. The latter implies that the expected gain a seller with signal s_1^* makes when she trades at $\rho(v,\eta) > v$ is exactly offset by trades at $\rho(v,\eta) < v$: these two regions are given by $[\underline{v}_1, v_1')$ and $(v_1', \overline{v}_1]$, respectively. Looking at the graph of $\alpha(v, 0.1)$ we can see that, as s_1^* increases, the distance between $\underline{v}_1$ and v_1' goes to zero implying that the set of trades with positive payoff shrinks to zero. Similarly, the distance between v_1' and $\overline{v}_1$ goes to zero when s_1^* decreases. Therefore, by the continuity of $\mathbb{E}(.|.)$ and $\alpha(., 0.1)$, we can find a unique triplet $(s_1^*, \underline{v}_1, \overline{v}_1)$ satisfying the conditions of *Proposition 1* and *Corollary 1*.

Before finishing the example, it is important to point out that, although $\rho(V,\eta)$ is the unique monotone equilibrium price, there are many possible equilibria associated with $\rho(V,\eta)$. Accordingly, I complete this example by characterizing one of the possible equilibrium bidding strategies (see bottom of *Figure 3*), namely the

[30] If $\mathbb{E}(V - \rho(V,\eta)|0) \geq 0$ all the mass of risk-neutral bids would be placed above $\rho(1,\eta)$ whereas it would be placed below $\rho(0,\eta)$ when $\mathbb{E}(V - \rho(V,\eta)|1) \leq 0$.

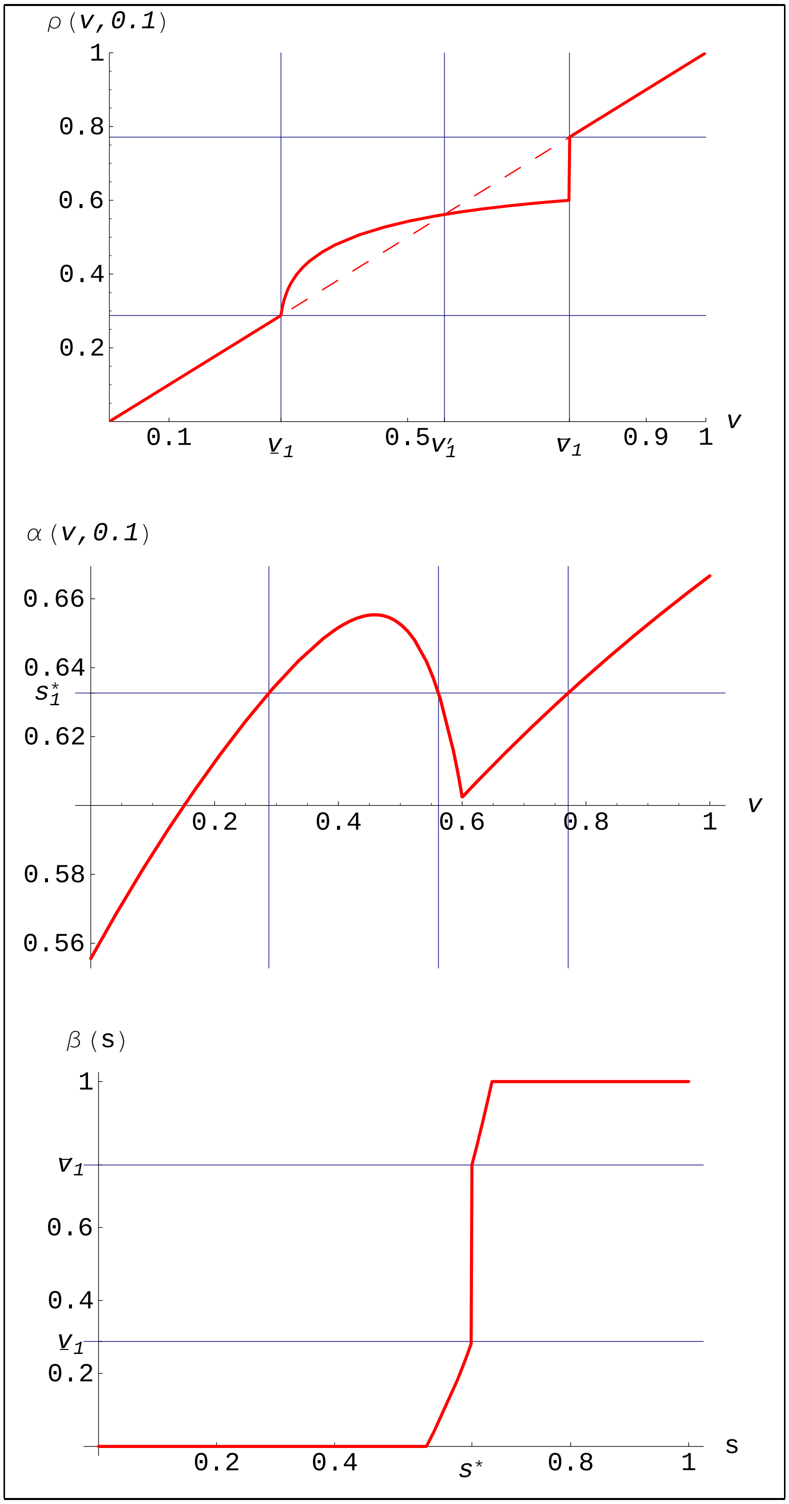

Figure 3: $\alpha(v, \eta)$ and $\rho(v, \eta)$ for $\gamma = 0.5$, $\eta = 0.1$

symmetric equilibrium $\beta(s,t) = \beta(s)$ given by

$$\beta(s) = \begin{cases} 0 & \text{if } s \in [0, \frac{1-\gamma}{1-\eta}] \\ \min v \text{ s.t. } \alpha(v,\eta) = s & \text{if } s \in (\frac{1-\gamma}{1-\eta}, s^*] \\ \max v \text{ s.t. } \alpha(v,\eta) = s & \text{if } s \in (s^*, \frac{1-\gamma-\eta}{1-\eta}] \\ 1 & \text{if } s \in (\frac{1-\gamma-\eta}{1-\eta}, 1] \end{cases} \tag{14}$$

4 An Empirical Test of Information Aggregation

Assuming the model laid out in the previous sections is a reasonable approximation of some existing asset markets,[31] *Proposition 1* suggests a way to identify empirically the intervals of asset values where $\rho(v,\eta) = v$. Furthermore, this identification should not require information on any of the parameters of the market, namely $\gamma, \eta, H(.|.), F(.|)$ and $G(.)$. The only identification restrictions (other than *Assumptions 1-4*) would be the monotonicity of equilibrium prices and the distribution of naïve bids having full support on the set of possible asset values.[32]

The heuristics of how to distinguish the set where prices are fully revealing from the set where $\rho(v,\eta) \neq v$ are rather simple. Recall that strategic bids are placed only in intervals of the price range where $\rho(v,\eta) = v$. Accordingly, the density of bids is higher in a small neighborhood of the observed price when it equals the unknown asset value than when value and price differ. That is, we should observe a discontinuous change in the density of bids at the boundaries $\{\underline{v}_k\}$ and $\{\overline{v}_k\}$ of non-revealing intervals. Specifically, as v increases, the density drops at $\{\underline{v}_k\}$ and jumps at $\{\overline{v}_k\}$, respectively. Hence, using only a series on prices and on a suitable measure of the size of the order flow around market prices one could statistically distinguish the two informational regimes: revealing versus non-revealing prices. Note also that by identifying the sets where prices differ from values we can establish a (not necessarily tight) upper bound on $|\rho(v,\eta) - v|$. Assume $[\rho(\underline{v}_k,\eta), \rho(\overline{v}_k,\eta)]$ is a non-revealing interval of prices. Then, $\rho(\underline{v}_k,\eta) = \underline{v}_k$ and $\rho(\overline{v}_k,\eta) = \overline{v}_k$. By monotonicity of prices, $|\rho(v,\eta) - v| < \rho(\overline{v}_k,\eta) - \rho(\underline{v}_k,\eta)$ for all $p \in [\rho(\underline{v}_k,\eta), \rho(\overline{v}_k,\eta)]$.

This approach can be of special relevance in some markets where the true value of the asset is never observed, and therefore the forecasting properties of prices are hard to assess. For instance, in markets where Arrow-Debreu securities are traded, the true value of the security (i.e. the probability of the state of nature in which

[31] Apart from prediction markets, other trading institutions with a double-auction format such as futures markets or stock exchanges could be suitable for this empirical approach as long as participants in those markets can reasonably characterized as either nonstrategic traders, who transact in these markets primarily driven by liquidity or similar considerations, and arbitrageurs (i.e. strategic agents), whose primary motive is to engage in speculative trading.

[32] Obviously, since this is an static model and most actual markets are dynamic, any empirical analysis would need some stationarity assumptions.

the security pays a dividend) is never observed. Only the realization of the state is observed.

5 A note on Equilibria in large CVDA

In the continuum agent economy analyzed in the previous sections agents use their private information to decide at which prices they are willing to trade. However, in markets with a finite number of agents, individual decisions have a non-negligible effect on prices. Therefore, it is natural to ask whether the characterization of prices given in *Proposition 1* is a good approximation of what happens in large but finite common value double auctions with naïve traders. That is, whether we can find a sequence of finite economies such that, as the number of traders goes to infinity, (i) equilibrium prices exist and, (ii) they "converge" to monotone equilibrium prices of the continuum agent CVDA.[33]

Although this analysis is beyond the scope of this paper it is worth discussing what might happen when agents can affect the price. The first thing to note is that when agents have the ability to affect prices zero (interim) expected payoffs may not be possible in equilibrium, because traders may have an incentive to deviate in order to push prices in their favor if the missed transactions for doing so involve zero profits. Hence, for equilibrium prices to converge to fully revealing prices they would need to balance vanishing positive expected payoffs with vanishing pivotal probabilities (i.e. the probability of a given agent to affect equilibrium prices). However, I conjecture that this is not possible, because the shares of naïve traders compatible with fully revealing prices ($\eta \leq \underline{\eta}$) are too low to create such arbitrage opportunities.[34] On the other hand, convergence in the partially and non-revealing regions should hold, given that in such scenario positive payoffs do not vanish in the limit.[35]

To provide some intuition, imagine that equilibrium prices in a large market look like those in the top graph of *Figure 4*. Unlike in the continuum agent market, there is a fundamental asymmetry in the preferences of buyers and sellers due to their incentive to affect prices in opposite directions. As a consequence, buyers and

[33] The appropriate notion of convergence would depend on the particular technique used in the asymptotic analysis. For instance, if the finite economies have a continuous bid space as in most analyses of private value double auctions, given the conditional independence of agents' signals, a natural notion is almost sure convergence in the probability space generated by the random vector of asset values and signal profiles. If an analysis à la Reny and Perry (2006) is required, where bids are restricted to be chosen from a discrete grid, a different notion of convergence would be needed.

[34] This is why for very low η only fully revealing prices are possible in any monotone equilibrium of the continuum agent CVDA.

[35] A way to get convergence to fully revealing prices could be to make the share of naïve traders converge from above to $\underline{\eta}$ at a lower rate than the rate at which pivotal probabilities vanish, instead of holding η constant.

sellers bid in different regions of the bidding space, i.e. now buyers (sellers) do not bid where expected prices lie above (below) values. Accordingly, prices below p_1 are slightly above expected values reflecting the overbidding of active sellers trying to increase the price while prices above p_2 are below expected values due to the underbidding of buyers. Note that no strategic seller (buyer) would bid very close to v_1, since he will only trade the object when prices are below (above) expected values. Therefore, only naïve bidders would bid in a neighborhood (p_1, p_2) of v_1. However, if η is too small, the mass of bids in (p_1, v_1) (respectively (v_1, p_2)) is not enough to bring expected prices above (below) values and prices would instead be like those in the bottom of *Figure 4*, which cannot be an equilibrium since buyers (sellers) bidding close to p_2 (p_1) would like to revise their bids.

6 Discussion and Concluding Remarks

Unlike REE models where the use of Walrasian equilibrium leaves price formation unspecified, double auctions provide insight into how individual trading decisions determine market prices so that information aggregation can be explicitly analyzed. In the pure common value market presented here, agents use their private information in their bidding strategies, despite (i) being price takers and (ii) their private information not adding any new information to the market. This stems from the fact that agents do not observe the price before deciding whether to trade or not. In contrast, in REE models the use of private information in large markets is often linked to risk aversion. This leads to a well-establish paradox associated to REE models (Hellwig (1980), Diamond and Verrecchia (1981), Grossman (1976, 1978)): if agents are risk-neutral, prices become fully revealing regardless the level of noise trade (modeled as random demand/supply).[36] Accordingly, agents dismiss their private information, which raises the question of how such information ends up reflected in the market price.

Hence, the analysis presented here provides an alternative theory of how and when prices aggregate information: even if no single agent can affect the price, private information is always useful to the individual trader since he can make inferences about the market price. This inference process is the mechanism by which individual information gets reflected into prices. Moreover, the usefulness of this information does not seem to depend on attitudes toward risk or on the size of the market. One could argue that the fact that information aggregation is driven by the use of private information to forecast prices is a more appealing explanation than information aggregation being driven solely by attitudes toward risk.

[36]For instance, in Hellwig (1980), as long as there is a non-vanishing share of risk-neutral agents, the unknown realization of the returns from the risky asset can be fully inferred from the price, for any finite *per capita* level of noise in the asset supply.

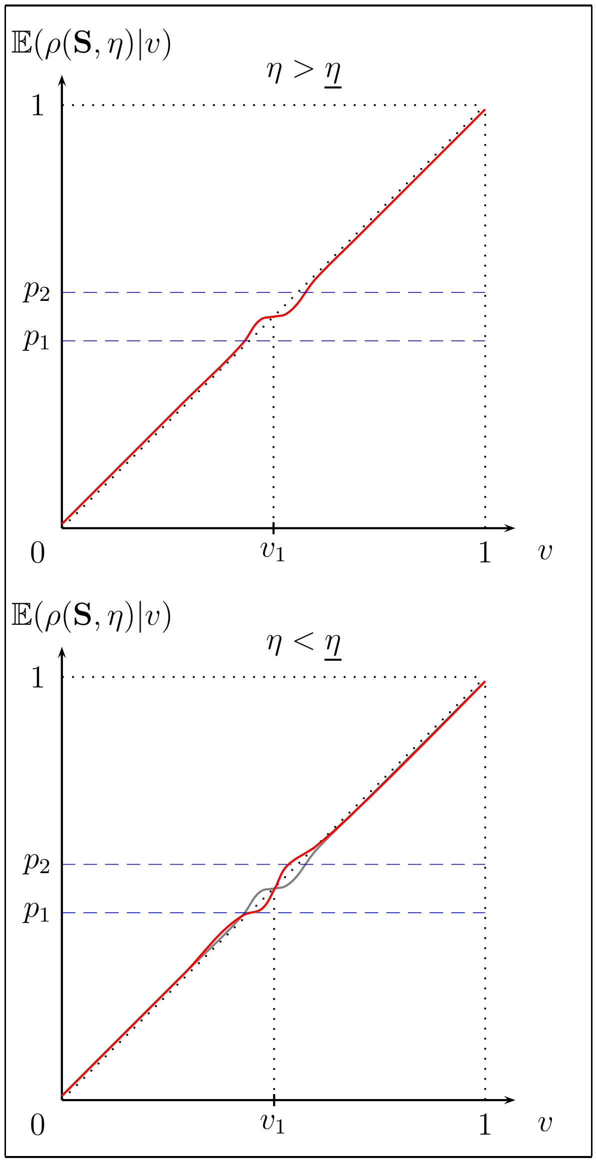

Figure 4: Prices in a finite common value double auction

The introduction of risk aversion should not change the implications of the model. Specifically, the key feature of strategic bidding behavior, namely that they do not bid in non-revealing areas of the equilibrium price function, still holds for risk averse agents. Accordingly, one should be able to prove results similar to those presented above for a population of strictly risk averse agents.[37] This would imply that risk aversion does not substantively affect the informational properties of prices. With regards to prediction markets, my motivating example, this means that their performance should not hinge upon eliciting risk neutral behavior.

It is important to point out that the assumption of common values is key to analyze information aggregation, given that they trigger the inference process that drives information into prices.[38] Thus, recent models analyzing prediction markets where agents fail to understand the existence of a common value component (Manski (2004), Gjerstad (2005), Wolfers and Zitzewitz (2006)) do not regard prices as information aggregators. Instead, they look at how close prices are to some statistic (e.g. the mean) of the distribution of agents' *interim* beliefs (i.e. beliefs based only on their privately held information). Their approach would be equivalent to having a share of naïve traders $\eta = 1$ when naïve traders bids are equal to the expected value of the asset conditional on their private signals. Not surprisingly, in such a case, prices are almost nowhere equal to asset values.[39]

From the analysis of the common value double auction, the main conclusion to extract is that, with price-taking behavior, prices aggregate information as long as the level of noise or liquidity trade, embodied by the presence of naïve traders, is small.[40] Otherwise, prices can be far from fully revealing. As the above example shows, the market price can be quite uninformative even for shares of naïve traders as low as 10%. This conclusion is at odds with some well-established views about the informational efficiency of markets. One such view is that a non-negligible share of *marginal* traders is enough to induce fully revealing prices (Wolfers and Zitzewitz (2004), Forsythe, Nelson, Neumann, and Wright (1992), Hellwig (1980)).[41] As it is

[37] Risk aversion breaks the symmetry of buyers and sellers' preferences and induces some strategic traders to abstain from trading. However, one should be able to get equilibrium prices similar to those given by *Proposition 1* by having two block-monotonic bid distributions, one for buyers and one for sellers. This reasoning applies to any population of traders with heterogeneous degrees of risk aversion, as long as the set of distinct attitudes toward risk is finite, so that the number of different block-monotonic distributions is also finite.

[38] With private values individual beliefs depend only on the individual's private information, thus learning the market price does not provide any additional insight to the trader about the value of the asset.

[39] One could argue that private information already incorporates common information, and markets simply attract well-informed individuals. In such a case, it would be interesting to model entry decisions as a function of the quality of private information. Neither the double auction model presented here nor existing models aimed at analyzing prediction markets deal with this issue.

[40] In the absence of price-taking behavior, one should expect prices being less informative.

[41] Marginal traders refer to those traders who react to their information and to changes in prices so as to induce prices to reflect available information. The definition of "marginal" trader

shown here, this is not true in the double auction. In the example provided, the minimum share of marginal traders compatible with fully revealing prices is about 98.5%. An alternative view is that, as long as there is a positive amount or noise or liquidity trade, prices cannot be fully revealing (Black (1986), Reny and Perry (2006), Kyle (1985, 1989)). In the double auction, this is not true either. When we account for the strategic incentives of traders, information aggregation is possible although it is quite sensitive to the amount of non-strategic traders in the market.

The analysis also provides insight into the qualitative behavior of strategic traders in double auctions. First, not surprisingly, low signal sellers and high signal buyers are more active in the market. That is, sellers with lower interim valuations and buyers with more positive views about the asset value will trade in a wider range of prices, compared to low signal buyers and high signal sellers. More interesting is that strategic traders restrict their asks and bids to areas where prices are fully revealing and, if that is not possible, they will bid outside the range of prices. This means that strategic traders are *perfect* price setters whenever possible, given that they set prices equal to values in the range of prices where they place their bids. This feature of the model provides the possibility to empirically distinguish between revealing and non-revealing areas without knowledge of the amount of naïve traders, their bidding behavior, or the distribution of private information.

I conclude with a word of caution. To induce price taking behavior I have analyzed a continuum agent market. Therefore, it remains to be seen if the results obtained carry through large but finite economies.

varies across models. For some authors it means traders not suffering from rationality "biases" (Forsythe, Nelson, Neumann, and Wright (1992)), whereas in REE models it means risk-neutral agents (Hellwig (1980)).

A Appendix: Proofs

A.1 Proofs of *Proposition 1* and *Corollary 1*

Proof of *Lemma 1.* Let $\rho(V, \eta)$ be the price function resulting from strategy profile $\beta(.,.)$, and assume buyer t and seller t' bid b when they receive signal s, i.e. $\beta(s, t) = \beta(s, t') = b$. If we subtract (7) from (6) we get

$$\pi^{buy}(s, t) = \pi^{sel}(s, t') + \mathbb{E}((V - \rho(V, \eta))|s). \tag{15}$$

Since the last term does not depend on b, a buyer and a seller receiving the same signal will have the same preference ranking over bids. ■

Let $\rho_+^{-1}(b, \eta) := \max\{v : \rho(v, \eta) = b\}$ and $\rho_-^{-1}(b, \eta) := \min\{v : \rho(v, \eta) = b\}$.

Lemma 2 (Strategic Bidding in monotone equilibria (I)) *Let $\beta(.,.)$ be a monotone equilibrium strategy profile and (v_1, v_2) be a non-degenerate set of asset values.*

(i) If $\rho(v, \eta) < v$ for all $v \in (v_1, v_2)$ and there is some trader t such that $\beta(s, t) < \rho(v_1, \eta)$ for some s, then there exists $v' < v_1$ such that $\rho(v, \eta) \geq v$ for all $v \in (\rho_-^{-1}(\beta(s, t), \eta), v')$, with strict inequality in a non-null subset.

(ii) If $\rho(v, \eta) > v$ for all $v \in (v_1, v_2)$ and there is some trader t such that $\beta(s, t) > \rho(v_2, \eta)$ for some s, then there exists $v' > v_2$ such that $\rho(v, \eta) \leq v$ for all $v \in (v', \rho_+^{-1}(\beta(s, t), \eta))$, with strict inequality in a non-null subset.

Proof of *Lemma 2.* Part (i): assume that $\rho(v, \eta) \leq v$ for all $v \in (\rho_-^{-1}(\beta(s, t), \eta), v_1)$. Then, given that $\mathbb{E}((V - \rho(V, \eta))1_{\{\rho(V,\eta)<v_2\}}|s) > 0$ for all s, a buyer would strictly prefer to bid v_2 than $\beta(s, t)$. Since preferences are symmetric, a seller would also prefer to bid v_2.[42]

A symmetric argument applies to (ii). ■

The following fact is needed for the proofs of *Lemma 3* and *Proposition 1.*

Fact 1 *Let Assumptions 1 and 2 be satisfied. If $\rho(v, \eta) > v$ for all $v \in [v_1, v_2)$ and $\rho(v, \eta) < v$ for all $v \in (v_2, v_3]$ with $\rho(., \eta)$ increasing, then for all $s \in (0, 1)$,*

(i) If $\mathbb{E}((V - \rho(V, \eta))1_{\{V\in[v_1,v_3]\}}|s) \leq 0$, then $\mathbb{E}((V - \rho(V, \eta))1_{\{V\in[v_1,v_3]\}}|s') < 0$ for all $s' < s$;

[42] Note that if b is an atom of the price distribution, $v' \leq \rho_+^{-1}(\beta(s, t), \eta)$ can only happen if the probability of getting the object is zero, i.e. $\lambda(\beta(s, t), v) = 0$ for all $v \in (\rho_-^{-1}(\beta(s, t), \eta), \rho_+^{-1}(\beta(s, t), \eta))$. Otherwise, a trader bidding at b would rather bid lower.

(ii) If $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s)\geq 0$*, then* $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s')>0$ *for all* $s'>s$*.*

Proof of *Fact 1.* Let $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s)\leq 0$. Thus,

$$\frac{1}{f(s)}\int_{v_1}^{v_2}(\rho(V,\eta)-V)f(s|v)g(v)dv \geq \frac{1}{f(s)}\int_{v_2}^{v_3}(V-\rho(V,\eta))f(s|v)g(v)dv. \quad (16)$$

By the strict monotone likelihood ratio of F (*Assumption 2*) we have that for all $s'<s$ and all $v'\in[v_1,v_2)$ and $v\in[v_2,v_3)$, $\frac{f(s'|v')}{f(s|v')}>\frac{f(s'|v)}{f(s|v)}$. Given this and the above inequality, we have that

$$\int_{v_1}^{v_2}(\rho(V,\eta)-V)f(s|v)\frac{f(s'|v)}{f(s|v)}g(v)dv > \int_{v_2}^{v_3}(V-\rho(V,\eta))f(s|v)\frac{f(s'|v)}{f(s|v)}g(v)dv. \quad (17)$$

Given that $f(s')>0$ for all s' by the full support of F and G (*Assumption 1*), (17) implies that $\mathbb{E}((\rho(V,\eta)-V)1_{\{V\in[v_1,v_2]\}}|s')>\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_2,v_3]\}}|s')$.

A symmetric argument applies to part (ii). ■

Lemma 3 (Strategic Bidding in monotone equilibria (II)) *In any monotone equilibrium, the mass of strategic traders submitting bids in* $\{\rho(v,\eta): v-\rho(v,\eta)\neq 0\}$ *is zero, except perhaps when there is a positive mass at the boundaries of the price range,* $\rho(0,\eta)$ *and* $\rho(1,\eta)$*, and there is complete rationing (*$1-\gamma=B(\rho(0,\eta)|v)$ *for all* $v\in[0,\rho_+^{-1}(0,\eta)]$*) and no rationing (*$1-\gamma=B\_(\rho(1,\eta)|v)$ *for all* $v\in[\rho_-^{-1}(1,\eta),1]$*), respectively.*

Proof of *Lemma 3.* By *Lemma 1*, I only need to look at a buyer's incentives. The proof goes along the following lines. First, I show that no strategic buyer is best-responding by bidding the interior of an interval of prices in which $\rho(v,\eta)\neq v$. Second, I show that if $\rho(.,\eta)$ is constant in an interval of values (i.e. the distribution of prices has an atom) a strategic buyer will only bid at the atom if she gets the object with probability zero or one, depending on whether the expected value of $\rho(V,\eta)-V$ at the atom is positive or negative, respectively. Otherwise, she would bid slightly above or below to avoid to either avoid trading or being rationed. Finally, I prove that these conditions cannot be satisfied at an atom in the interior of the price range. Therefore, the only possibility left for a strategic buyer to bid in $\{\rho(v,\eta): v-\rho(v,\eta)\neq 0\}$ is to bid at the boundaries, with the condition that she does not trade almost surely when she bids $\rho(0,\eta)$ and that she trades with probability one when bidding $\rho(1,\eta)$.

Assume, that a buyer bids in an interval $(\rho(v_1,\eta),\rho(v_2,\eta))$ where $v>\rho(v,\eta)$ and $\rho(v,\eta)$ is strictly increasing a.e. in (v_1,v_2), i.e. there is no atom in (v_1,v_2). In this case, she prefers to bid v_2 to any bid $b \in (\rho(v_1, \eta), \rho(v_2, \eta))$, given that her payoff increases by $\mathbb{E}((V - \rho(V, \eta))1_{\{\rho(V,\eta) \in (b, \rho(v_2, \eta))\}}|s)$, which is strictly positive for all s. Now assume that $v < \rho(v, \eta)$ in (v_1, v_2). Then, a buyer would prefer to bid below $\rho(v_1, \eta)$ given that $\mathbb{E}((V - \rho(V, \eta))1_{\{\rho(V,\eta) \in (\rho(v_1, \eta), b)\}}|s) < 0$ for all s.

Now assume there is an atom at b. If there is a positive mass of strategic bids at b, with $\rho(v, \eta) = b$ on some interval (v_1, v_2), a buyer with signal s might bid at $b \in (0, 1)$ under one of these conditions: (i) $\mathbb{E}((V - b)1_{\{\rho(V,\eta)=b\}}|s) = 0$; (ii) $\mathbb{E}((V - b)1_{\{\rho(V,\eta)=b\}}|s) > 0$ with $\lambda(b, v) = 1$ for all $v \in (v_1, v_2)$ (i.e. no rationing); (iii) $\mathbb{E}((V - b)1_{\{\rho(V,\eta)=b\}}|s) < 0$ with $\lambda(b, v) = 0$ for all $v \in (v_1, v_2)$ (i.e. no trading when $\rho(., \eta) = b$).

In case (i), she is indifferent between bidding slightly above or below b. However, *Fact 1* implies that there can be at most one signal satisfying (i).[43] Therefore the mass of bids at b due to (i) is zero. In (ii), unless she gets the object with probability one ($\lambda(b, v) = 1$) she would rather bid above b. Finally, in (iii) she may bid at b only if the probability of getting the object is zero ($\lambda(b, v) = 0$). Since in each of the latter two cases $\lambda(b, .)$ is required to be zero or one in the whole interval (v_1, v_2), there cannot be two traders bidding at b with distinct signals satisfying (ii) and (iii), respectively. Accordingly, all the mass of strategic bidders bidding at b either satisfy (ii) or (iii).

Now I show that (ii) and (iii) can only happen when $b = \rho(1, \eta)$ and $b = \rho(0, \eta)$, respectively.

Assume (ii) is satisfied for all bidders bidding at b and let $\underline{s}$ be the lowest signal associated to b. Accordingly, a trader receiving $\underline{s}$ bids optimally at b if

$$\mathbb{E}((V - \rho(V, \eta))1_{\{p \leq b\}}|\underline{s}) \geq 0, \tag{18}$$

and

$$\mathbb{E}((V - \rho(V, \eta))1_{\{p > b\}}|\underline{s}) \leq 0. \tag{19}$$

By *Lemma 2*, we can apply *Fact 1* to (18) and (19).[44] Hence, all strategic traders with signals above $\underline{s}$ will bid at or above b (assuming $\lambda(b, v) = 1$). Likewise, given (19) and the fact that $\mathbb{E}((V - b)1_{\{\rho(V,\eta)=b\}}|\underline{s}) \geq 0$, all strategic traders with $s < \underline{s}$ will bid strictly below b.

For $\lambda(b, v) = 1$ we need the mass of sellers bidding strictly less than b be equal to the mass of buyers bidding at b or above. That is, for all $v \in (v_1, v_2)$,

$$\gamma[\eta H(b|v) + (1 - \eta)F(\underline{s}|v)] = (1 - \gamma)[\eta(1 - H(b|v)) + (1 - \eta)(1 - F(\underline{s}|v))]. \tag{20}$$

[43] For (i) to hold $v < b$ in the lower part of (v_1, v_2) and $v > b$ in the upper part of the interval, so that *Fact 1* applies.

[44] By the linearity of expectations, the conclusions of *Fact 1* also apply to a succession of intervals satisfying the conditions in the lemma.

Given that $B\_(b|v) = \eta H(b|v) + (1-\eta)F(\underline{s}|v)$, the above expression is satisfied when $B\_(b|v) = 1-\gamma$ for all $v \in (v_1, v_2)$.

Now assume that $b < \rho(1,\eta)$, i.e. $v_2 < 1$ and $\rho(v,\eta) > b$ for all $v > v_2$. For that to happen we need $B(b|v) < 1-\gamma$ for all $v > v_2$. But this implies, by the continuity of $B(b|v)$ and $B\_(b|v)$, that there exists $v' < v_2$ such that $B\_(b|v) < B(b|v) \leq 1-\gamma$ for all $v \geq v'$, which contradicts that $\lambda(b,v) = 1$ for all $v \in (v_1, v_2)$. Hence, (ii) is only possible in equilibrium if $b = \rho(1,\eta)$ and $v_2 = 1$.

A symmetric argument applies when (iii) is satisfied for almost all strategic traders bidding at b. In this case, the mass of sellers bidding at or below b needs to be equal to the mass of buyers bidding strictly above b for $\lambda(b,v) = 0$. This requires that $B(b|v) = 1-\gamma$ for all $v \in (v_1, v_2)$. By the continuity of $B(b|v)$ and $B\_(b|v)$, $b = \rho(0,\eta)$ and $v_1 = 0$, otherwise there would be a subset of (v_1, v_2) for which $1-\gamma \leq B\_(b|v) < B(b|v)$, contradicting that $\lambda(b,v) = 0$ for all $v \in (v_1, v_2)$.

Finally, if $\rho(1,\eta)$ is not an atom, the probability of rationing at $\rho(1,\eta)$ is zero and a buyer bidding $\rho(1,\eta)$ will always trade. In this case there can exist a positive mass of strategic bids at $\rho(1,\eta) < 1$, since any buyer bidding $\rho(1,\eta) < 1$ is indifferent between any two bids in $[\rho(1,\eta), 1]$. A symmetric argument can be made for bids at $\rho(0,\eta) > 0$. ■

Lemma 3 allows for the possibility of having strategic bids placed at an atom, at $\rho(0,\eta)$ or at $\rho(1,\eta)$, of the price distribution if either sellers or buyers bidding at the atom trade with probability one, respectively. However, as the next lemma shows, atoms can only occur for very particular naïve share and bid distributions.

Lemma 4 (No atoms) *In any monotone equilibrium if there exists $v_1 < v_2$ such that $\rho(v,\eta) = b$ on (v_1, v_2) then*

(a) $\mathbb{E}((V-\rho(v,\eta))1_{\{V<v_2\}}|s) \geq 0$ for all s, and $H(\rho(v,\eta)|v) = \frac{1-\gamma}{\eta}$ for all $v \leq v_2$;

(b) $\mathbb{E}((V-\rho(v,\eta))1_{\{V<v_2\}}|s) \leq 0$ for all s, and $H(\rho(v,\eta)|v) = \frac{\eta-\gamma}{\eta}$ for all $v \geq v_1$.

Lemma 4 basically states that atoms in the price distribution are solely created by naïve traders, and that very special circumstances need to occur: the share of naïve bids is very high compared to γ (or to $1-\gamma$); naïve traders completely determine prices at the low (high) end of the price range, with those prices being low (high) enough so that they do not encourage strategic traders to bid below (above) the atom; and the distribution of naïve bids is independent of asset values in the interval of values associated with the atom.[45]

[45] An example of equilibrium prices being constant in some interval of values is given by a high enough presence of random traders bidding uniformly in $[0,1]$. In this case, $H(b|v) = b$ for all v. Hence, if η is high enough so that $\mathbb{E}(V|0) \geq b = \frac{1-\gamma}{\eta}$, then $\rho(v,\eta) = \frac{1-\gamma}{\eta}$ for all v, with all strategic traders bidding at or above $\frac{1-\gamma}{\eta}$. In this case, all strategic buyers and no strategic seller engage in trade.

Proof of *Lemma 4.* Assume there is an interval (v_1, v_2) such that $\rho(v, \eta) = b$ for all $v \in (v_1, v_2)$. By *Lemma 2* and *Fact 1*, if there exists a trader with signal s bidding below (above) b then it is optimal for all traders with signals below (above) s to also bid below (above) b. Accordingly, let $\underline{s} \in [0, 1]$ be the highest signal associated with bids below b, and $\overline{s} \geq \underline{s}$ the lowest signal associated with bids above b. Since the distribution of naïve bids is atomless (*Assumption 3*), this implies that

$$B\_(b|v) = \eta H(b|v) + (1 - \eta)F(\underline{s}|v),$$

and

$$B(b|v) = \eta H(b|v) + (1 - \eta)F(\overline{s}|v).$$

There are two possible cases, depending on whether a positive mass of strategic bids is placed at b or not, i.e. whether $\underline{s} < \overline{s}$ or $\underline{s} = \overline{s}$.

If there is no positive mass of strategic bids at b, we have that $B(b|v) = B\_(b|v) = 1 - \gamma$ for all $v \in (v_1, v_2)$. Since $F(s|v)$ is strictly decreasing in v for all $s \in (0, 1)$ and $H(b|v)$ is non-increasing in v for all $b \in [0, 1]$, $B\_(b|v) = 1 - \gamma$ for all $v \in (v_1, v_2)$ only if $\underline{s} = 0$ or $\underline{s} = 1$.

a) $\underline{s} = 0$: in this case $H(b|v) = \frac{1-\gamma}{\eta}$ for all $v \in (v_1, v_2)$. But then, we need $\mathbb{E}((V - \rho(V, \eta))1_{\{\rho(V,\eta) \leq b\}}|s) = \mathbb{E}((V - \rho(V, \eta))1_{\{V < v_2\}}|s) \geq 0$ for all s, otherwise some strategic traders would rather bid below b. Finally, prices below b are completely determined by naïve bids, since no strategic trader bids below b, i.e. $H(\rho(v, \eta)|v) = \frac{1-\gamma}{\eta}$ for all $v \leq v_1$.[46]

b) $\underline{s} = 1$: in this case $H(b|v) = \frac{\eta-\gamma}{\eta}$ for all $v \in (v_1, v_2)$. In addition, we need $\mathbb{E}((V - \rho(V, \eta))1_{\{V < v_2\}}|s) \leq 0$ for all s. Since no strategic trader bids above b, prices above b are given by $H(\rho(v, \eta)|v) = \frac{\eta-\gamma}{\eta}$ for all $v \geq v_2$.

If there is a positive mass of strategic bids at b, *Lemma 3* applies, requiring either that $B\_(b|v) = 1 - \gamma$ or $B(b|v) = 1 - \gamma$. The former requires $\underline{s} = 0$ or $\underline{s} = 1$, while the latter can be possible only if $\overline{s} = 0$ or $\overline{s} = 1$. Therefore, they reduce to the same conditions on $H(.|.)$ and $\mathbb{E}((V - \rho(V, \eta))1_{\{\rho(V,\eta) \leq b\}}|s)$.

■

Proof of *Proposition 1.* By the monotonicity of $\rho(., \eta)$ and *Lemma 3*, all the mass of strategic bids in $(\rho(0, \eta), \rho(1, \eta))$ is placed in a countable collection of disjoint intervals in which $\rho(v, \eta) = v$. Let $\mathcal{V}$ be the complement of such set in $[0, 1]$. Thus,

[46]This is possible in principle given that $H(.|.)$ is increasing in its first argument and decreasing in its second argument.

$\mathcal{V} \supseteq \{v : \rho(v,\eta) \neq v\}$ by *Lemma 3*. Assume $\mathcal{V}$ is non-empty, otherwise *Proposition 1* holds trivially.

Denote $B^*(.|.)$ the cdf of strategic bids and assume that $B^*(.|.)$ is atomless.[47] Accordingly, $B^*_-(.|v) = B^*(.|v)$ for all $v \in [\rho(0,\eta), \rho(1,\eta)]$ and $\mathcal{V}$ can be expressed, without loss of generality, as the countable union of disjoint closed intervals $[\underline{v}_k, \overline{v}_k]$, such that $\rho(v,\eta)$ is given by

$$H(\rho(v,\eta)|v) = \frac{1-\gamma-(1-\eta)B^*(\rho(\underline{v}_k,\eta)|v)}{\eta} \text{ for all } v \in [\underline{v}_k, \overline{v}_k]. \tag{21}$$

Further assume that prices are not a.e. equal to values in $[\underline{v}_k, \overline{v}_k]$. Otherwise, redefine $\mathcal{V}$ not to include such interval.

Notice that $B^*(\rho(\underline{v}_k,\eta)|v) = B^*(\rho(\overline{v}_k,\eta)|v)$ by *Lemma* 3 for all k, including non-revealing intervals with $\underline{v}_1 = 0$ (i.e. when $\rho(0,\eta) > 0$) and $\overline{v}_K = 1$ ($\rho(1,\eta) < 1$). Hence, we just need to show that $B^*(\rho(\underline{v}_k,\eta)|v) = F(s^*_k|v)$, for all k and all $v \in [\underline{v}_k, \overline{v}_k]$, with s^*_k satisfying (9)-(11).

By *Lemma 2* and the fact that $\rho(0,\eta) > 0$ and $\rho(1,\eta) < 1$, there exist v'_k, v''_k with $\underline{v}_k < v'_k \leq v''_k < \overline{v}_k$ such that $\rho(v,\eta) \geq v$ a.e. in $[\underline{v}_k, v'_k]$ with strict inequality in a non-null subset, and $\rho(v,\eta) \leq v$ a.e. in $[v''_k, \overline{v}_k]$ with strict inequality in a non-null subset.[48] But this implies that, if bidding in $[\overline{v}_{k-1}, \rho(\underline{v}_k)]$ is optimal for a seller with signal s, i.e. $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s) \leq 0$, then $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s') < 0$ for all $s' < s$ by *Fact 1*.[49] Hence, bidding above $\rho(\overline{v}_k)$ is strictly dominated by bidding in $[\overline{v}_{k-1}, \rho(\underline{v}_k)]$ for all sellers with $s' < s$. A symmetric argument can be used for all $s' > s$ when it is optimal for a seller with signal s to bid in $[\rho(\overline{v}_k), \underline{v}_{k+1}]$. By *Lemma 1* the same applies for a buyer. Therefore, $B^*(\rho(\underline{v}_k)|v) = F(s^*_k|v)$ for some signal $s^*_k > 0$. Moreover, s^*_k needs to satisfy (9) if $\underline{v}_k > 0$ and (10) whenever $\overline{v}_k < 1$, given that $\rho(v,\eta) = v$ in $(\overline{v}_{k-1}, \underline{v}_k)$ and in $(\overline{v}_k, \underline{v}_{k+1})$ and that $H(.|.)$, $F(.|.)$ are atomless distributions. Finally, condition (11) is just the equilibrium condition for a seller with $s \leq s^*_k$ ($s > s^*_k$) to optimally bid below $\rho(\underline{v}_k)$ (above $\rho(\overline{v}_k)$), which also implies that $s^*_{k-1} < s^*_k$ for all $k > 1$.

Now assume that $B^*(.|.)$ has an atom. Since $H(.|.)$ does not have atoms in $(\rho(0,\eta), \rho(1,\eta))$, neither can $B^*(.|.)$. An atom of $B^*(.|.)$ at $b \in (\rho(0,\eta), \rho(1,\eta))$ would imply that $B_-(b|b) < B(b|b)$, creating an atom of the price distribution at b, which leads to a non-revealing interval where strategic bids are placed, a contradiction of *Lemma 3*. Therefore, $B^*(.|.)$ can have an atom only in $\{\rho(0,\eta), \rho(1,\eta)\}$.

[47] This implies that $\rho(0,\eta) > 0$ and $\rho(1,\eta) < 1$, given that $H(0) = 0 < 1-\gamma$ and $H(1) = 1 > 1-\gamma$ by *Assumption 3*.

[48] In what follows, I use the convention, $\overline{v}_0 = 0$ and $\underline{v}_{K+1} = 1$.

[49] By the linearity of expectations, this is also true when there are non-revealing intervals above $\overline{v}_k$ satisfying *Lemma 2*, or subintervals in $[v'_k, v''_k]$ with $\rho(v,\eta) > (<)v$. For instance, if $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s) + \sum_{k'>k}\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\underline{v}'_k,\overline{v}'_k)\}}|s) \leq 0$ for some s, with $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s) < 0$, then these inequalities hold strictly for all $s' < s$.

If there is an atom in $B^*(.|.)$ at $\rho(0,\eta)$, the price distribution may or may not have an atom at $\rho(0,\eta)$. If the price distribution has an atom at $\rho(0,\eta)$, by *Lemma 4*, $B^*(\rho(0,\eta)|v) = 1$ for all v and $H(\rho(0,\eta)|v) = \frac{\eta-\gamma}{\eta}$ for all $v \leq \rho_+^{-1}(\rho(0,\eta),\eta)$. Accordingly, $B^*(\rho(0,\eta)|v) = F(1|v)$ and (8) is satisfied. If the price distribution does not have an atom at $\rho(0,\eta)$, $\rho(0,\eta)$ is given by (21), i.e.

$$H(\rho(0,\eta)|0) = \frac{1-\gamma-(1-\eta)B^*(\rho(0,\eta)|0)}{\eta}. \tag{22}$$

Hence, if a non-revealing interval starts at $\rho(0,\eta)$ (i.e. $\underline{v}_1 = 0$), *Lemma 2* applies to the interval $[0,\overline{v}_1]$ and, by *Fact 1*, there exists a signal $s_1^* > 0$ satisfying (11) such that $B^*(\rho(0,\eta)|v) = F(s_1^*|v)$.

Finally, assume $B^*(.|.)$ has an atom at $\rho(1,\eta)$. If the price distribution has an atom at $\rho(1,\eta)$, $B_-^*(\rho(1,\eta)|v) = 0$ for all v and $H(\rho(1,\eta)|v) = \frac{1-\gamma}{\eta}$ for all $v \geq \rho_-^{-1}(\rho(0,\eta))$ by *Lemma 4*. Thus, $B_-^*(\rho(1,\eta)|v) = F(0|v)$ and (8) is also satisfied. If the price distribution does not have an atom at $\rho(1,\eta)$, $\rho(1,\eta)$ is given by

$$H(\rho(1,\eta)|1) = \frac{1-\gamma-(1-\eta)B_-^*(\rho(1,\eta)|1)}{\eta}. \tag{23}$$

Therefore, if a non-revealing interval ends at $\rho(1,\eta)$, *Lemma 2* applies to the interval $[\overline{v}_K, 1]$ and, by *Fact 1*, there exists a signal $s_K^* < 1$ satisfying (11) such that $B^*(\rho(0,\eta)|v) = F(s_1^*|v)$.

■

Proof of *Corollary 1*. First, note that by *Lemma 2*, all intervals $(\rho(\underline{v}_k,\eta),\rho(\overline{v}_k,\eta))$ with strategic bids below and above them satisfy the first part of *Corollary 1*, i.e. prices are above values in the lower portion of the interval $(\underline{v}_k,\overline{v}_k)$ and below values in the upper portion. When there are no strategic bids placed below $(\rho(\underline{v}_k,\eta),\rho(\overline{v}_k,\eta))$ we have that $\underline{v}_k = 0$ given that $H(0) = 0 < 1-\gamma$ implies $\rho(0,\eta) > 0$. If there are strategic bids above this interval part (ii) of *Lemma 2* applies. Finally, no strategic bids above $(\rho(\underline{v}_k,\eta),\rho(\overline{v}_k,\eta))$ lead to $\overline{v}_k = 1$ and $\rho(\overline{v}_k,\eta) < 1$. In addition, if there are strategic bids below part (i) of *Lemma 2* applies. Hence, these two cases also satisfy the first part of the corollary.

This also applies when there is an atom at $(\underline{v}_k,\overline{v}_k)$. Since atoms can only happen when there are no bids either below or above the atom (*Lemma 4*), we are in one of the above cases.

Regarding the second part of *Corollary 1*, assume there is a non-revealing interval $(\underline{v}_k,\overline{v}_k)$ that can be partitioned into two subintervals, $(\underline{v}_k,\hat{v}]$ and $(\hat{v},\overline{v}_k)$, such that, in each of them, prices are above values in the lower portion of the subinterval and below values in the upper portion with $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\hat{v},\overline{v}_k)\}}|s_k^*) < 0$. In such case, $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\overline{v}_k,\hat{v}]\}}|s_k^*) > 0$ and a strategic seller bidding below $\rho(\underline{v}_k,\eta)$

with signal close to s_k^* would rather deviate and bid $\hat{v}$ since, by doing so, he can avoid negative payoffs from trading when $v \in (\overline{v}_k, \hat{v}]$. By symmetry of preferences a buyer with the same signal would also deviate. ■

A.2 Proof of *Proposition 2*

I provide a series of technical lemmas and facts related to the quantile function $\alpha(., \eta)$, which are used in the proof of *Proposition 2*. They show that $\alpha(., \eta)$ is increasing for small values of η, non-monotonic for intermediate values and not well-defined and decreasing when η is high enough. As shown below, prices must equal values everywhere in the first case, and cannot be revealing in areas where $\alpha(., \eta)$ is either not defined or decreasing.

In what follows, D_i represents the partial derivative with respect to the ith argument.

Lemma 5 *If Assumptions 1-4 are satisfied the following statements are true:*

(i) $\alpha(0, \eta)$ *is well-defined for* $\eta < \gamma$*, strictly positive and increasing in* η*;* $\alpha(1, \eta)$ *is well-defined for* $\eta < 1 - \gamma$*, strictly less than one and decreasing in* η*.*

(ii) If $D_1\alpha(v, \eta) < 0$ *then* $D_1\alpha(v, \eta') < 0$ *for all* $\eta' > \eta$ *for which* $\alpha(v, \eta)$ *is well defined.*

(iii) There exists $\underline{\eta} \in (0, \min\{\gamma, 1-\gamma\})$ *such that* $\alpha(., \eta)$ *is well-defined and strictly increasing for all* $\eta < \underline{\eta}$*, and it is non-monotonic or decreasing for all* $\eta > \underline{\eta}$ *in the subset of values where it is well-defined.*

(iv) If $H'(v) > 0$ *for all* v *such that* $H(v) = 1 - \gamma$*, there exists* $\overline{\eta} \in [\underline{\eta}, 1)$ *such that* $\alpha(., \eta)$ *is decreasing whenever it is well-defined for all* $\eta > \overline{\eta}$*.*

Proof of Lemma 5.

Part (i): since $H(0) = 0$, $\alpha(0, \eta) = F^{-1}(\frac{1-\gamma}{1-\eta}|0)$, which is well-defined if $\eta < \gamma$. Since $F(.|v)$ has full support for all v and $\frac{1-\gamma}{1-\eta}$ is increasing in η, $\alpha(0, \eta)$ is increasing in η. Similarly, $H(1) = 1$ so $s(1, \eta) = F^{-1}(\frac{1-\gamma-\eta}{1-\eta}|0)$ is well-defined for $\eta < 1 - \gamma$ and decreasing in η.

Part (ii): by the a.e. smoothness of H and F (*Assumptions 1* and *3*), $\alpha(v, \eta)$ is a.e. differentiable. I differentiate both sides of (12) to obtain $D_1\alpha(v, \eta)$:

$$D_1\alpha(v, \eta) = -\frac{\frac{\eta}{1-\eta}H'(v) + D_2F(\alpha(v, \eta)|v)}{f(\alpha(v, \eta)|v)}. \tag{24}$$

Note that $f(.|.) > 0$ by the full support assumption. In addition, $D_2F(.|.) < 0$ by strict MLRP. Therefore, for $D_1\alpha(v,\eta) < 0$ we need the numerator of (24) to be negative, i.e.

$$\frac{\eta}{1-\eta}H'(v) + D_2F(\alpha(v,\eta)|v) > 0. \tag{25}$$

Thus, if we show that (25) implies

$$\frac{\partial}{\partial\eta}\left[\frac{\eta}{1-\eta}H'(v) + D_2F(\alpha(v,\eta)|v)\right] \geq 0,$$

which is equivalent to

$$\frac{H'(v)}{(1-\eta)^2} \geq -D_2f(\alpha(v,\eta)|v)D_2\alpha(v,\eta), \tag{26}$$

then we would have shown that if the numerator of (24) is negative, it becomes more negative as η grows. This will suffice to prove part (ii) of the lemma.

Given that $D_2\alpha(v,\eta) = \frac{1-\gamma-H(v)}{(1-\eta)^2 f(\alpha(v,\eta)|v)}$, (26) can be expressed as

$$H'(v) \geq -(1-\gamma-H(v))\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}. \tag{27}$$

Therefore, we need to prove that (25) implies (27). By the strict MLRP of f, $\frac{F(s|v)}{f(s|v)}$ is decreasing in v and $\frac{1-F(s|v)}{f(s|v)}$ is increasing in v for all s. Thus,

$$\frac{\partial}{\partial v}\left[\frac{F(s|v)}{f(s|v)}\right] = \frac{f(s|v)D_2F(s|v) - D_2f(s|v)F(s|v)}{f^2(s|v)} \leq 0, \tag{28}$$

and

$$\frac{\partial}{\partial v}\left[\frac{1-F(s|v)}{f(s|v)}\right] = \frac{-f(s|v)D_2F(s|v) - D_2f(s|v)(1-F(s|v))}{f^2(s|v)} \geq 0. \tag{29}$$

Consequently, (28) and (29) imply that $\frac{D_2f(s|v)}{f(s|v)} \in \left[\frac{D_2F(s|v)}{F(s|v)}, \frac{-D_2F(s|v)}{1-F(s|v)}\right]$ for all $s \in (0,1)$ and all v.[50]

We need to consider two possible cases: $H(v) < 1-\gamma$ and $H(v) > 1-\gamma$.[51]

1. $H(v) < 1-\gamma$: if we divide both sides of (25) by $F(\alpha(v,\eta)|v)$,[52] we obtain

[50] By the full support assumption $F(s|v) \in (0,1)$ for all $s \in (0,1)$ and the bounds on $\frac{D_2f(s|v)}{f(s|v)}$ are well defined.

[51] If $H(v) = 1-\gamma$, (27) is satisfied given that $H'(v) > 0$ is needed for (25) to hold.

[52] We can do so since $F(\alpha(v,\eta)|v) > 0$ whenever $H(v) < 1-\gamma$. Assume $F(\alpha(v,\eta)|v) = 0$ otherwise. Then $1-\gamma = \eta H(v)$ and $H(v) > 1-\gamma$, a contradiction.

$$\frac{\eta}{1-\eta}\frac{H'(v)}{F(\alpha(v,\eta)|v)} > -\frac{D_2F(\alpha(v,\eta)|v)}{F(\alpha(v,\eta)|v)} \geq -\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.$$

Substituting $F(\alpha(v,\eta)|v) = \frac{1-\gamma-\eta H(v)}{1-\eta}$ in the above expression and multiplying both sides by $(1-\gamma-H(v)) > 0$ we get

$$H'(v)\frac{\eta(1-\gamma-H(v))}{1-\gamma-\eta H(v)} > -(1-\gamma-H(v))\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}. \tag{30}$$

Since $H(v) < 1-\gamma$, $\gamma \in (0,1)$ and $\eta \in (0,1]$,[53] $\frac{\eta(1-\gamma-H(v))}{1-\gamma-\eta H(v)}$ is strictly positive and less than one. Hence, (30) implies (27) given that $H'(v) > 0$ by (25).

2. $H(v) > 1-\gamma$: two subcases need to be considered. If $D_2f(\alpha(v,\eta)|v) \leq 0$ the right-hand side of (27) is non-positive. Thus, (27) is satisfied for all v such that $H'(v) > 0$ and all η. When $D_2f(\alpha(v,\eta)|v) > 0$, by dividing both sides of (25) by $1-F(\alpha(v,\eta)|v)$ we get[54]

$$\frac{\eta}{1-\eta}\frac{H'(v)}{1-F(\alpha(v,\eta)|v)} > -\frac{D_2F(\alpha(v,\eta)|v)}{1-F(\alpha(v,\eta)|v)} \geq \frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.$$

Substituting $F(\alpha(v,\eta)|v)$ and rearranging terms, the above inequality becomes

$$\begin{aligned} \eta H'(v) &> (1-\eta+\eta H(v)-(1-\gamma))\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)} \\ &\geq -(1-\gamma-H(v))\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}. \end{aligned} \tag{31}$$

The second inequality holds because $1-\eta+\eta H(v) \geq H(v)$ and, therefore, $(1-\eta+\eta H(v)-(1-\gamma)) \geq -(1-\gamma-H(v)) > 0$.[55] Since $\eta \in (0,1]$, (31) implies (27).

Part (iii): first, note that $\alpha(.,\eta)$ is well-defined in $[0,1]$ iff $\eta \leq \eta_\gamma := \min\{\gamma, 1-\gamma\}$, given that $\frac{1-\gamma-\eta H(v)}{1-\eta} \in [\frac{1-\gamma}{1-\eta}, \frac{1-\gamma-\eta}{1-\eta}]$ for all $v \in [0,1]$.

[53] If (25) holds then $\eta > 0$.

[54] Note that $F(\alpha(v,\eta)|v) < 1$ whenever $H(v) > 1-\gamma$. If $F(\alpha(v,\eta)|v) = 1$ then $1-\gamma-\eta H(v) = 1-\eta$, which can only hold if $H(v) < 1-\gamma$ given that $1-\eta+\eta H(v) = H(v)+(1-\eta)(1-H(v))$.

[55] To see why notice that $1-\eta+\eta H(v) = (1-\eta)(1-H(v))+H(v)$, which is at least $H(v)$ given that $\eta \in (0,1]$ and $H(v) \in (1-\gamma, 1]$.

As $\eta \to 0$, $\frac{1-\gamma-\eta H(v)}{1-\eta} \to 1-\gamma$. Thus, $\lim_{\eta\to 0} \alpha(v,\eta) = F^{-1}(1-\gamma|v)\ \forall v$. By *Assumptions 1-2*, $F^{-1}(1-\gamma|.)$ is well-defined and strictly increasing in $[0,1]$. By the continuity of F and H, $\alpha(.,\eta)$ is continuous and strictly increasing in $[0,1]$ for all η sufficiently small. This takes care of $\underline{\eta}$ being strictly positive. We need to show that there exists $\underline{\eta} < \eta_\gamma$ such that for all $\eta < \underline{\eta}$, $\alpha(.,\eta)$ is everywhere increasing, and for all $\eta > \underline{\eta}$, there exists some $v \in [0,1]$ such that $\alpha(v,\eta)$ is well-defined with $D_1\alpha(v,\eta) < 0$.

Since $H(0) = 0$ and $H(1) = 1$, $H'(v) > 0$ in a set of asset values with positive Lebesgue measure, so for high enough η inequality (25) is satisfied for some v. Assume for the moment that such η is lower than η_γ. By part (ii) of the lemma, if (25) is satisfied for η and v it will also be satisfied for all $\eta' > \eta$. Accordingly, if $D_1\alpha(v,\eta) > 0$, then $D_1\alpha(v,\eta'') > 0$ for all $\eta'' < \eta$. Therefore, given that $D_1\alpha(v;.)$ is well-defined and continuous in $[0,\eta_\gamma]$ for all v, $\underline{\eta}$ exists and is given by the highest η such that $D_1\alpha(v,\eta) \geq 0$ for all v, i.e.

$$\underline{\eta} := \sup_{\eta} \left\{ \eta \in (0,1) : \eta \leq -\frac{D_2F(F^{-1}(\frac{1-\gamma-\eta H(v)}{1-\eta}|v)|v)}{H'(v) - D_2F(F^{-1}(\frac{1-\gamma-\eta H(v)}{1-\eta}|v)|v)} \ \ \forall v \right\}. \tag{32}$$

It remains to be shown that there is a $\eta < \eta_\gamma$ satisfying (25) for some v. Assume otherwise that $\underline{\eta}$, the highest η for which there is no v satisfying (25), is greater than η_γ.

1. If $\underline{\eta} \geq \gamma$, then $\frac{1-\gamma}{1-\underline{\eta}} > 1$. Since $\frac{1-\gamma-\eta}{1-\underline{\eta}} < 1$ and $H(.)$ is continuous, there exists an interval of values $[\underline{v},\overline{v}]$ such that $\frac{1-\gamma-\underline{\eta}H(\underline{v})}{1-\underline{\eta}} = 1$ and $\frac{1-\gamma-\underline{\eta}H(v)}{1-\underline{\eta}}$ is strictly decreasing in v for all $v \in [\underline{v},\overline{v}]$. Hence, by the full support assumption, $\alpha(\underline{v},\underline{\eta}) = 1$ and $\alpha(v,\underline{\eta}) < 1$ for all $v \in (\underline{v},\overline{v}]$, which implies that $D_1\alpha(v,\underline{\eta}) < 0$ for some v, contradicting that (25) is not satisfied by $\underline{\eta}$.

2. If $\underline{\eta} \geq 1-\gamma$, we have that $\frac{1-\gamma-\underline{\eta}}{1-\underline{\eta}} < 0$. Since $\frac{1-\gamma}{1-\underline{\eta}} > 0$, there exists an interval of values $[\underline{v}',\overline{v}']$ such that $\frac{1-\gamma-\underline{\eta}H(\overline{v}')}{1-\underline{\eta}} = 0$ and $\frac{1-\gamma-\underline{\eta}H(v)}{1-\underline{\eta}}$ is strictly decreasing in v for all $v \in [\underline{v}',\overline{v}']$. Hence, by the full support assumption, $\alpha(\overline{v}',\underline{\eta}) = 0$ and $\alpha(v,\underline{\eta}) > 0$ for all $v \in [\underline{v}',\overline{v}')$, which implies that $D_1\alpha(v,\underline{\eta}) < 0$ for some v, a contradiction.

<u>Part (iv)</u>: note that, as $\eta \to 1$, $\frac{1-\gamma-\eta H(v)}{1-\gamma} \to \infty$ for all v such that $H(v) < 1-\gamma$ and $\frac{1-\gamma-\eta H(v)}{1-\gamma} \to -\infty$ for all v such that $H(v) > 1-\gamma$. Therefore, for high enough η, $\alpha(.,\eta)$ is only well-defined in a small neighborhood of all v such that $H(v) = 1-\gamma$. If for any such v we have that $H'(v) > 0$ then $\alpha(.,\eta)$ will be decreasing in such

neighborhood.[56] By part (ii) of the lemma, if $\alpha(.,\eta)$ is decreasing, it is decreasing for all $\eta' > \eta$. ■

Fact 2 *Let $\mathcal{S}$ be a measurable subset of $[0,1]$ and $s \in (0,1)$ be such that $\mathbb{P}(\mathcal{S}|v) = F(s|v)$ for some $v \in [0,1)$. Then, $D_2\mathbb{P}(\mathcal{S}|v) \geq D_2F(s|v)$.*

Proof of Fact 2. Assume $\mathcal{S} \cap [s,1]$ is a non-null set, otherwise $\mathbb{P}(\mathcal{S}|v) = F(s|v)$ for all v by the full support of $F(.|v)$ for all v. Since $\mathbb{P}(\mathcal{S}|v) = F(s|v) = \mathbb{P}([0,s]|v)$, we have that

$$\mathbb{P}([s,1] \cap \mathcal{S}|v) = \mathbb{P}([0,s] \setminus \mathcal{S}|v).$$

By the strict MLRP of $F(.|.)$, the left hand side is strictly greater than the right-hand side for all $v' > v$. Thus, $D_2\big[\mathbb{P}(\mathcal{S}|v) - \mathbb{P}([0,s]|v)\big] \geq 0$. ■

Lemma 6 *If $\alpha(.,\eta)$ is strictly decreasing in some interval $[v_1, v_2]$ then any monotone equilibrium price $\rho(.,\eta)$ satisfies $\rho(v,\eta) \neq v$ a.e. in $[v_1, v_2]$.*

Proof of Lemma 6. Assume $\rho(v,\eta) = v$ and $\rho(v',\eta) = v'$ for some $v' > v$ with $\alpha(v,\eta) > \alpha(v',\eta)$. Accordingly, if the mass of strategic bids below v is given by bidders with signals in $[0, \alpha(v,\eta)]$, then the mass of bids below $v' > v$ is strictly smaller than the mass of bids below v, a contradiction. Hence, it must be that there is an alternative, well-defined distribution of strategic bids $B^a(.|.)$ such that $B^a(v|v) = \frac{1-\gamma-\eta H(v)}{1-\eta}$ for all $v \in [v_1, v_2]$. Since $\alpha(.,\eta)$ is decreasing in that interval, we have that $\frac{d}{dv}B^a(v|v) = -\frac{\eta}{1-\eta}H'(v) < D_2F(\alpha(v,\eta)|v)$ (see inequality (25)).

Denoting $\beta^a(s,t)$ the bid of strategic trader t when she receives signal s, we have that

$$B^a(v|v) = \int_{\mathcal{T}}\int_0^1 1_{\{\beta^a(s,t)\leq v\}} f(s|v)\,ds\,d\mu = \int_{\mathcal{T}} \mathbb{P}(\mathcal{S}^a(v,t)|v)\,d\mu,$$

where $\mathcal{S}^a(v,t) = \{s \in [0,1] : \beta^a(s,t) < v\}$.

By *Fact 2*, $D_2\mathbb{P}(\mathcal{S}^a(v,t)|v) \geq D_2F(s^a(v,t)|v)$ with $s^a(v,t)$ being the signal such that $\mathbb{P}(\mathcal{S}^a(v,t)|v) = F(s^a(v,t)|v)$. Accordingly, given that $D_1B^a(v|v) \geq 0$,

$$\frac{d}{dv}B^a(v|v) = D_1B^a(v|v) + D_2B^a(v|v) \geq \int_{\mathcal{T}} D_2F(s^a(v,t)|v)\,d\mu.$$

[56] In this case there is a unique v such that $H(v) = 1-\gamma$. Assume otherwise that there are two such values v, v' such that $H'(v), H'(v') > 0$. Since $H(0) = 0$ and $H(1) = 1$, by the continuity of $H(.)$, there would have to be a value $v'' \in$ such that $H(v'') = 1-\gamma$ and $H'(v'') < 0$.

Therefore, it is enough to show that $\int_{\mathcal{T}} D_2 F(s^a(v,t)|v)d\mu \geq D_2 F(\alpha(v,\eta)|v)$ whenever $\int_{\mathcal{T}} F(s^a(v,t)|v)d\mu = F(\alpha(v,\eta)|v)$ in order to prove that there is no $B^a(.|.)$ leading to revealing prices in $[v_1, v_2]$.

By the strict MLRP we have that, for all $v \in [v_1, v_2]$ and all $v' > v$,

$$
\begin{aligned}
0 &= \int_{\mathcal{T}} F(s^a(v,t)|v)d\mu - F(\alpha(v,\eta)|v) = \\
&= \int_{\mathcal{T}} \int_{\alpha(v,\eta)}^{s^a(v,t)\vee\alpha(v,\eta)} f(x|v)dxd\mu - \int_{\mathcal{T}} \int_{\alpha(v,\eta)\wedge s^a(v,t)}^{\alpha(v,\eta)} f(x|v)dxd\mu \\
&\leq \int_{\mathcal{T}} \int_{\alpha(v,\eta)}^{s^a(v,t)\vee\alpha(v,\eta)} f(x|v)\frac{f(x|v')}{f(x|v)}dxd\mu - \int_{\mathcal{T}} \int_{\alpha(v,\eta)\wedge s^a(v,t)}^{\alpha(v,\eta)} f(x|v)\frac{f(x|v')}{f(x|v)}dxd\mu \\
&= \int_{\mathcal{T}} F(s^a(v,t)|v')d\mu - F(\alpha(v,\eta)|v').
\end{aligned}
$$

Therefore, $\int_{\mathcal{T}} D_2 F(s^a(v,t)|v)d\mu \geq D_2 F(\alpha(v,\eta)|v)$, which implies $\frac{d}{dv}B^a(v|v) \geq D_2 F(\alpha(v,\eta)|v)$, a contradiction. ∎

Proof of Proposition 2.

The proof is divided into two cases, depending on the value of η. For $\eta \in [0, \underline{\eta}]$, where $\underline{\eta} > 0$ is given by (32), I show that prices are necessarily fully revealing; whereas when $\eta > \underline{\eta}$ prices cannot be fully revealing. In the latter case, I provide an algorithm to find monotone equilibrium prices satisfying *Proposition 1* and *Corollary 1* and show that they exist and are unique. After that, I show that, except for a very particular class of naïve distributions, there exists $\overline{\eta} < 1$ such that for all $\eta \geq \overline{\eta}$ strategic bids are confined outside the range of equilibrium prices, implying that $\mathcal{V} = [0,1]$.

Before turning into these cases, a prerequisite for existence is that any equilibrium prices satisfying *Proposition 1* and *Corollary 1* are in fact increasing. This is guaranteed if any block-monotonic distribution of strategic bids leads to market clearing prices that are increasing. According to *Proposition 1*, market prices in non-revealing intervals are given by

$$H(p|v) = \frac{1 - \gamma - (1-\eta)F(s_k^*|v)}{\eta}. \tag{33}$$

Given any $\eta \in (0,1)$, the right-hand side of this expression is constant for $s_k^* \in \{0,1\}$ and strictly increasing in v for $s_k^* \in (0,1)$. Hence, when $H(.|.)$ satisfies *Assumption 4*, the resulting price is increasing in v.

Now I turn into the two cases to be considered, $\eta \in [0, \underline{\eta}]$ and $\eta \in (\underline{\eta}, 1]$.

Case 1: ($\eta \in [0, \underline{\eta}]$). The function $\alpha(v, \eta)$ given by (12) is the quantile of the signal distribution leading to strategic bids at or below v such that $\rho(v, \eta) = v$. By *Lemma 5*, $\alpha(., \eta)$ is increasing for all $\eta < \underline{\eta}$. This implies that there exists a fully revealing equilibrium for all $\eta < \underline{\eta}$, with the distribution of strategic bids satisfying $B^*(v|v) = F(\alpha(v, \eta)|v)$. Since prices are fully revealing and agents cannot affect the price, no strategic trader has an incentive to deviate and, hence, any profile of bidding strategies yielding B^* constitutes a BNE. One such profile is given by $\beta(s, t) = \beta(s)$ for all t, with

$$\beta(s) = \begin{cases} 0 & \text{if } s \in [0, \alpha(0, \eta)] \\ v \text{ s.t. } \alpha(v, \eta) = s & \text{if } s \in (\alpha(0, \eta), \alpha(1, \eta)) \\ 1 & \text{if } s \in [\alpha(1, \eta), 1] \end{cases} \tag{34}$$

This takes care of existence of monotone equilibrium for $\eta \in [0, \underline{\eta}]$.

Regarding uniqueness of monotone equilibrium prices, assume there exists a monotone equilibrium with $\rho(v, \eta) \neq v$ a.e. in $(\underline{v}_1, \overline{v}_1)$ with $\underline{v}_1 < \overline{v}_1$ for some $\eta \leq \underline{\eta}$. If $\rho(\underline{v}_1, \eta) < \rho(\overline{v}_1, \eta)$, by *Lemma 3*, the mass of strategic bids placed in $[\rho(\underline{v}_1, \eta), \rho(\overline{v}_1, \eta)]$ is zero. Since the distribution of strategic bids is block-monotonic (*Proposition 1*), all strategic traders with signals below (above) some signal s_1^* bid below $\rho(\underline{v}_1, \eta)$ (above $\rho(\overline{v}_1, \eta)$). However, given that $\alpha(., \eta)$ is increasing, we have that $s_1^* > \alpha(\underline{v}_1, \eta)$ and/or $s_1^* < \alpha(\overline{v}_1, \eta)$.[57] When $s_1^* > \alpha(\underline{v}_1, \eta)$ then $\rho(\underline{v}_1, \eta) < \underline{v}_1$ if $\underline{v}_1 > 0$ or $\rho(v, \eta) = 0$ in $[0, v')$ for some $v' > 0$ if $\underline{v}_1 = 0$, contradicting *Corollary 1*. On the other hand, if $s_1^* < \alpha(\overline{v}_1, \eta)$ then $\rho(\overline{v}_1, \eta) > \overline{v}_1$ if $\overline{v}_1 < 1$ or $\rho(v, \eta) = 1$ in $(v'', 1]$ for some $v'' < 1$ if $\overline{v}_1 = 1$, which again violates *Corollary 1*. Therefore, the only possibility left is that $\rho(v_1, \eta) = \rho(v_2, \eta)$, i.e. there exist an atom in the distribution of prices. But, according to *Lemma 4*, this can only happens when $\eta \geq \min\{\gamma, 1 - \gamma\}$, i.e. when $\eta > \underline{\eta}$.

Hence, when $\eta \in [0, \underline{\eta}]$ any monotone equilibrium price satisfies $\rho(v, \eta) = v$.

Case 2: ($\eta \in (\underline{\eta}, 1]$). By part (ii) of *Lemma 5* $\alpha(., \eta)$ is either non-monotonic or decreasing. Hence, prices cannot be fully revealing, given *Lemma 6*. The following algorithm identifies the values $\{\underline{v}_k\}_{k=1}^K$, $\{\overline{v}_k\}_{k=1}^K$ and signals $\{s_k^*\}_{k=1}^K$ that satisfy the conditions of *Proposition 1* and *Corollary 1*, which characterize equilibrium prices. Then I show that these values and signals always exist and are unique. Finally, I provide bidding strategies that implement equilibrium prices.

The steps of the algorithm are:

1. Find asset values $\{v_i^m\}_{i=1}^I$ and $\{v_i^M\}_{i=1}^{I'}$ at which $\alpha(., \eta)$ reaches a local minimum and a local maximum, respectively. If $\alpha(., \eta)$ is not well-defined in an

[57] Note that $\alpha(., \eta)$ is strictly increasing for $\eta < \underline{\eta}$. If $\eta = \underline{\eta}$ and $\alpha(., \underline{\eta})$ is constant in $[\underline{v}_1, \overline{v}_1]$ then $s_1^* = \alpha(\underline{v}_1, \underline{\eta})$ would involve $\rho(v, \underline{\eta}) = v$ in $[\underline{v}_1, \overline{v}_1]$. Hence, one of these inequalities still needs to hold for $\rho(v, \underline{\eta}) \neq v$ a.e. in $[\underline{v}_1, \overline{v}_1]$.

interval (v', v'') with $\alpha(v', \eta)$ or $\alpha(v'', \eta) \in \{0, 1\}$, let v' be the "unique" local maximum in that interval when $\alpha(v'', \eta) = 1$ and v'' be the "unique" local minimum when $\alpha(v', \eta) = 0$.[58] Let $v_0^m = 0$ and $v_{I'+1}^M = 1$.[59]

2. For each interval $\{[v_{i-j}^m, v_{i+1}^M]\}_{i=1}^{I-1+j}$, with $j = 0$ if $v_1^m = 0$ and $j = 1$ if $v_1^M = 0$, find signal values $\{s_i\}_{i=1}^{I-1+j}$ such that, when $\rho(v, \eta)$ satisfies $1 - \gamma = \eta H(\rho(v, \eta)|v) + (1 - \eta)F(s_i|v)$, are given by

$$s_i = \begin{cases} 0 & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [v_{i-j}^m, v_{i-j+1}^m]\}}|0) > 0, \\ 1 & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [v_i^M, v_{i+1}^M]\}}|1) < 0, \\ s & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [\underline{\nu}_i(s), \overline{\nu}_i(s)]\}}|s) = 0, \end{cases} \tag{35}$$

where $\underline{\nu}_i(s), \overline{\nu}_i(s)$ are respectively given by

$$\underline{\nu}_i(s) = \begin{cases} v_{i-j}^m & \textit{if } \alpha(v_{i-j}^m, \eta) > s, \\ v \in [v_{i-j}^m, v_i^M] \ \textit{s.t.} \ \alpha(v, \eta) = s & \textit{otherwise,} \end{cases} \tag{36}$$

and

$$\overline{\nu}_i(s) = \begin{cases} v_{i+1}^M & \textit{if } \alpha(v_{i+1}^M, \eta) < s, \\ v \in [v_{i-j+1}^m, v_{i+1}^M] \ \textit{s.t.} \ \alpha(v, \eta) = s & \textit{otherwise.} \end{cases} \tag{37}$$

3. If $s_i > s_{i+1}$ merge intervals $[v_{i-j}^m, v_{i+1}^M]$ and $[v_{i+1-j}^m, v_{i+2}^M]$ and redefine $s_i = s_i'$ and $\overline{\nu}_i(s_i') = \overline{\nu}_{i+1}(s_i')$, with s_i' given by

$$s_i' = \begin{cases} 0 & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [v_{i-j}^m, v_{i-j+2}^m]\}}|0) > 0, \\ 1 & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [v_i^M, v_{i+2}^M]\}}|1) < 0, \\ s & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [\underline{\nu}_i(s), \overline{\nu}_{i+1}(s)]\}}|s) = 0. \end{cases} \tag{38}$$

Repeat this step until $s_i \leq s_{i+1}$ for $i = 1, ..., K$, with K being the new number of intervals.

4. Define $s_k^* = s_k$, $\underline{v}_k = \underline{\nu}_k(s_k)$ and $\overline{v}_k = \overline{\nu}_k(s_k)$, $k = 1, ..., K$.

Several things are worth noting. First, each interval $[v_{i-j}^m, v_{i+1}^M]$ contains v_{i-j+1}^m and v_i^M. Thus, $\alpha(., \eta)$ is increasing in $(v_{i-j}^m, v_i^M) \cup (v_{i-j+1}^m, v_{i+1}^M)$ and decreasing

[58] Note that when $\alpha(v'', \eta) = 1$ either $\alpha(v', \eta) = 1$ or it is not well-defined. Similarly, when $\alpha(v', \eta) = 0$, when $\alpha(v'', \eta) = 0$ or it is not well-defined.

[59] By the continuity of $\alpha(., \eta)$, $v_i^m < v_i^M$ for all i if $\alpha(0, \eta)$ is a local minimum, and $v_i^M < v_i^m$ for all i if $\alpha(0, \eta)$ is a local maximum.

in (v_i^M, v_{i-j+1}^m).[60] This implies that $s_i \in \left[\alpha(v_{i-j+1}^m,\eta), \max\{\alpha(v_i^M,\eta), \alpha(v_{i+1}^M,\eta)\}\right]$. Assume otherwise that $s_i < \alpha(v_{i-j+1}^m,\eta) < 1$. Then $\rho(v,\eta) > v$ in $[\underline{\nu}_i(s_i), \overline{\nu}_i(s_i)]$ since $\alpha(v,\eta)$ is above s_i in $[\underline{\nu}_i(s_i), v_{i+1}^M(s_i)]$, leading to $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[\underline{\nu}_i(s_i),\overline{\nu}_i(s_i)]\}}|s_i) < 0$ when $s_i < 1$, which violates (35). Given these bounds on s_i, there exists a unique value $\nu_i'(s_i) \in (v_i^M, v_{i-j+1}^m)$ such that $\alpha(\nu_i'(s_i),\eta) = s_i$. Accordingly, $\rho(v,\eta) > v$ in $(\underline{\nu}_i(s_i), \nu_i'(s_i))$ and $\rho(v,\eta) < v$ in $(\nu_i'(s_i), \overline{\nu}_i(s_i))$.[61]

Second, $\underline{\nu}_i(.)$ and $\overline{\nu}_i(.)$ are increasing, while $v_i'(.)$ is decreasing. By the continuity assumptions and *Fact 1*, each tuple $(s_i, \underline{\nu}_i(s_i), \overline{\nu}_i(s_i))$ exists and it is unique. To see why, note that as s_i grows the interval where prices are above values $(\underline{\nu}_i(s_i), \nu_i'(s_i))$ shrinks while $(\nu_i'(s_i), \overline{\nu}_i(s_i))$ grows. Furthermore, as s_i grows the probability mass (conditional on s_i) associated to $(\nu_i'(s_i), \overline{\nu}_i(s_i))$ grows relative to the mass associated to $(\underline{\nu}_i(s_i), \nu_i'(s_i))$, by the MLRP of $F(.|s_i)$. Therefore, there is a unique signal s_i (which in turn uniquely determines $\underline{\nu}_i(s_i)$ and $\overline{\nu}_i(s_i)$) satisfying (35).

Third, when two adjacent intervals with signals s_i, s_{i+1} are merged (step 3 of the algorithm), the new pivotal signal s_i' lies in (s_{i+1}, s_i). Thus, any subinterval of $[\underline{\nu}_i(s_i'), \overline{\nu}_{i+1}(s_i')]$ with $\rho(v,\eta) < v$ is preceded by a subinterval with $\rho(v,\eta) > v$, which means that we can apply the same existence and uniqueness argument to the tuple $(s_i', \underline{\nu}_i(s_i'), \overline{\nu}_{i+1}(s_i'))$.

Finally, $\alpha(.,\eta)$ is increasing on $[0, \underline{\nu}_1(s_1)]$, $[\overline{\nu}_i(s_i), \underline{\nu}_{i+1}(s_i)]$ and on $[\underline{\nu}_K(s_K), 1]$. That is, it is increasing in $[0,1] \setminus \bigcup_k [\underline{v}_k, \overline{v}_k]$, which enables prices to be fully revealing in such set (*Lemma 6*).

Given all these facts, (35)-(38) imply that $\{(s_k^*, \underline{v}_k, \overline{v}_k)\}$ satisfy (9)-(11). Moreover, prices given by (8) are monotonic and satisfy *Corollary 1.*

Since this algorithm provides a unique solution, we need to show that a collection $\{(s_h', \underline{v}_h', \overline{v}_h')\}$ not satisfying (35)-(38) violates (9)-(11) or *Corollary 1.*

Assume that there is a collection $\{(s_h', \underline{v}_h', \overline{v}_h')\}$ satisfying *Proposition 1.*

If $s_h' \in (0,1)$ then $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[\underline{v}_h',\overline{v}_h']\}}|s_h') = 0$ by (11). In addition, (9)-(10) and *Corollary 1* require that $\alpha(\underline{v}_h',\eta) \geq s_h'$ with equality when $\underline{v}_h' \in (0,1)$ and $\alpha(\overline{v}_h',\eta) \leq s_h'$ with equality when $\underline{v}_h' \in (0,1)$. *Corollary 1* further requires $\alpha(v,\eta)$ to be increasing at $v = \underline{v}_h', \overline{v}_h'$ whenever $\alpha(v,\eta) = s_h'$. All these conditions imply that $\underline{v}_h \in [v_{i-j}^m, v_i^M]$ and $\overline{v}_h \in [v_{l-j}^m, v_l^M]$ for some i, l with $i < l$. But then, if $i = l+1$, $(s_h', \underline{v}_h', \overline{v}_h') = (s_i, \underline{\nu}_i, \overline{\nu}_i)$ given (35)-(38). On the other hand, if $i < l+1$ let s_k, $k = i, ..., l$, be the signals given by (35). If $s_k < s_{k+1}$ for all k

[60] Note that for $i=0$, $v_{i-j}^m = v_i^M$ when $v_1^M = 0$, and $v_{i-j+1}^m = v_{i+1}^M$ for $i=1$ when $v_K^m = 1$.

[61] This is also true when $s_i \in \{0,1\}$. Given (36)-(37), $s_i = 0$ implies that $\underline{\nu}_i = v_{i-j}^m$ and $\frac{1-\gamma-\eta H(v)}{1-\eta} < 0$ in some interval (v', v_i^m) (otherwise (35) would be violated), which leads to $\overline{\nu}_i = v_i^m$ (according to step 1 of the algorithm, v_i^m is the upper bound of the interval of values where $\alpha(.,\eta)$ is not well-defined). The latter implies that $\rho(v,\eta) < v$ in $(v', \overline{\nu}_i)$. Since $\alpha(.,\eta)$ is either increasing in (v_{i-j}^m, v_i^M) or above 0 when $v_{i-j}^m = 0$ (part (i) of *Lemma 5*), $\alpha(v,\eta) > 0$ (and thus $\rho(v,\eta) > v$) in $(\underline{\nu}_i, v')$. Similarly, $s_i = 1$ implies that $\underline{\nu}_i = v_i^M$ and $\frac{1-\gamma-\eta H(v)}{1-\eta} > 1$ in some interval (v_i^M, v'), which means that $\overline{\nu}_i = v_{i+1}^M$. Hence, $\rho(v,\eta) > v$ in (v_i^M, v') and $\rho(v,\eta) < v$ in (v', v_{i+1}^M).

then $s'_h \in (s_i, s_l)$ for $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{v}'_h,\overline{v}'_h\}}]|s'_h) = 0$ to hold. But this implies that $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{v}'_h,\overline{\nu}_i(s'_h)\}}]|s'_h) > 0$, which by *Fact 1* means that $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\overline{\nu}_i(s'_h),\overline{v}'_h\}}]|s'_h) < 0$. Thus, a strategic trader receiving s'_h would rather bid $\overline{\nu}_i(s'_h)$ than bid below $\underline{v}'_h$, contradicting that $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ correspond to equilibrium prices. Assume then that there exists some $i \le h \le l$ such that $s_h \ge s_{h+1}$. In such case, abusing notation, let $\{s_{i'}\}$ denote the new collection of signals given by (38) after merging intervals $[v^m_{h-j}, v^M_{h+1}]$ and $[v^m_{h+1-j}, v^M_{h+2}]$. If $s_{i'} < s_{i'+1}$ for some i' in the new collection of signals, we again have that $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\overline{\nu}_{i'}(s'_h),\overline{v}'_h\}}]|s'_h) < 0$, which leads to a profitable deviation by a trader receiving signal s'_h. By using this argument iteratively, we arrive at the conclusion that applying the algorithm to the subcollection of intervals $[v^m_{i-j}, v^M_{i+1}]$ that are included in $[\underline{v}'_h, \overline{v}'_h]$ we obtain a unique signal s_i such that $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{\nu}_i(s_i),\overline{\nu}_{i+1}(s_i)]\}}|s_i) = 0$, with $\underline{\nu}_i(s_i) \in [v^m_{i-j}, v^M_i]$ and $\overline{\nu}_i(s_i) \in [v^m_{l-j}, v^M_l]$. But then, as shown above, $(s_i, \underline{\nu}_i(s_i), \overline{\nu}_i(s_i))$ is the unique tuple satisfying (36)-(38), which are equivalent to (9)-(11), and that is compatible with equilibrium behavior by strategic traders. Hence, $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ cannot part of a characterization of equilibrium prices if $(s'_h, \underline{v}'_h, \overline{v}'_h) \neq (s_i, \underline{\nu}_i(s_i), \overline{\nu}_i(s_i))$.

If $s'_h = 0$ then $\underline{v}'_h = 0$ by part (i) of *Lemma 5* and $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[0,\overline{v}'_h\}}]|0) \ge 0$ by (11). In addition, *Corollary 1* requires that $\rho(v,\eta) < v$ in the upper part of $[0, \overline{v}'_h)$, which means that $\overline{v}'_h \in [v^m_{i-j}, v^m_{i-j+1}]$ for some $i = 1, ..., I - 1 + j$. But this can only happen if $\frac{1-\gamma-\eta H(v)}{1-\eta} < 0$ in some interval (v', v^m_{i-j+1}). Thus, $\overline{v}'_h = v^m_{i-j+1}$, otherwise $\overline{v}'_1$ would not satisfy (10). We need to consider two cases. If $i = 1$ we have that the unique triplet satisfying these conditions is $(s_1, \underline{v}_1, \overline{v}_1)$ as defined by the above algorithm. If $i > 1$ and $s_l < s_{l+1}$ for all $l < i$, then $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{\nu}_{l+1}(s_{l+1}),\overline{\nu}_{l+1}(s_{l+1})]\}}|s_{l+1}) = 0$ with $\underline{\nu}_{l+1}(s_{l+1}) < v^m_{i-j+1}$ (otherwise $\rho(v,\eta) < v$ a.e. in $[\underline{\nu}_{l+1}(s_{l+1}), \overline{\nu}_{l+1}(s_{l+1})]$) and a trader receiving a signal in (s_l, s_{l+1}) would rather deviate and bid $\underline{\nu}_i(s_{l+1})$. Therefore, $s_l > s_{l+1} = 0$ for some $l < i$. Using iterative merging we arrive at the conclusion that either $(s'_h, \underline{v}'_h, \overline{v}'_h) = (s^*_1, \underline{v}_1, \overline{v}_1)$ or that $(s'_h, \underline{v}'_h, \overline{v}'_h)$ violates *Proposition 1.*

Finally, when $s'_h = 1$ we have that $\overline{v}'_h = 1$ by part (i) of *Lemma 5* and $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{v}'_h\}}]|1) \le 0$ by (11). The latter implies that $\frac{1-\gamma-\eta H(v)}{1-\eta} > 1$ in an interval $(v^M_{i-j'}, v')$ for some $i = 1, ..., I'$ with $j' = 0$ if $v^M_{I'} < 1$ and $j = 1$ otherwise, whereas *Corollary 1* requires that $\rho(v,\eta) > v$ in the lower part of $(\underline{v}'_h, 1]$. Hence, $\underline{v}'_h = v^M_i$ by (9). When $i = I'$, $(s_K, \underline{v}_K, \overline{v}_K)$ is the only triplet satisfying the above conditions. If $i < I'$ it has to be that $s_l > s_{l+1} = 0$ for some $l \ge i$, otherwise a trader receiving a signal in (s_l, s_{l+1}) would rather deviate and bid $\overline{\nu}_{l+1}(s_{l+1}) > v^M_i$, given *Fact 1* and that $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{\nu}_{l+1}(s_l),\overline{\nu}_l(s_l)]\}}|s_l) = 0$. By the usual merging argument it has to be that either $(s'_h, \underline{v}'_h, \overline{v}'_h) = (s^*_K, \underline{v}_K, \overline{v}_K)$ or that it violates *Proposition 1.*

This completes the proof that a collection $\{(s^*_k, \underline{v}_k, \overline{v}_k)\}$ satisfying (9)-(11) exists and it is unique. We just need to provide an example of bidding strategies yielding such equilibrium prices. The following symmetric strategies implement equilibrium prices characterized by $\{(s_k^*, \underline{v}_k, \overline{v}_k)\}$ and (8):

$$\beta(s) = \begin{cases} 0 & \text{if } s \in [0, s_1^*] \\ v \in [\overline{v}_k, \underline{v}_{k+1}] \text{ s.t. } \alpha(v, \eta) = s & \text{if } s \in (s_k^*, s_{k+1}^*] \\ 1 & \text{if } s \in (s_K^*, 1] \end{cases} \tag{39}$$

To complete the proof of *Proposition 2*, we need to show that there exists $\overline{\eta}$ such that $\mathcal{V} = [0, 1]$ for all $\eta \geq \overline{\eta}$. By *Lemma 6*, revealing prices can only exist for values such that $\alpha(., \eta)$ is increasing. In addition, by *Lemma 5*, once $\alpha(., \eta)$ is decreasing at v it is decreasing for all $\eta' > \eta$. Therefore, if there exists a share $\overline{\eta}$ such that $\alpha(., \eta)$ is either decreasing or not well-defined, it will also be so for all $\eta > \overline{\eta}$. In this context, the mass of bids at $[0, \rho(0, \eta)]$ (resp. $[\rho(1, \eta), 1]$) is given by the mass of signals $s \leq s_1^*$ ($s > s_1^*$), where

$$s_1^* = \begin{cases} 0 & \textit{if } \mathbb{E}(V - \rho(V, \eta)|s) > 0 \ \ \forall s, \\ 1 & \textit{if } \mathbb{E}(V - \rho(V, \eta)|s) < 0 \ \ \forall s, \\ s \ \textit{s.t.} \ \mathbb{E}(V - \rho(V, \eta)|s) = 0 & \textit{otherwise,} \end{cases} \tag{40}$$

with $\rho(v, \eta)$ satisfying $1 - \gamma = \eta H(\rho(v, \eta)|v) + (1 - \eta) F(s_i|v)$. The signal s_1^* exists and it is unique as shown above.

Note that, by *Lemma 5*, if for any v such that $H(v) = 1 - \gamma$ we have that $H'(v) > 0$, then there exists $\overline{\eta} < 1$ such that $\alpha(., \eta)$ is decreasing for all $\eta > \underline{\eta}$, leading to nowhere revealing prices (*Lemma 6*).

If, however, $H'(v) < 0$ for at least a value v satisfying $H(v) = 1 - \gamma$, a revealing region may exist around v for all $\eta < 1$. To see why, note that for any such value there are two values v' and v'' with $v' < v < v''$ such that $H(v') = H(v'') = 1 - \gamma$ and $H(v'), H(v'') > 0$. Accordingly, for η close to one, $\alpha(., \eta)$ is decreasing in a neighborhood of v' and v'' and increasing in a neighborhood of v, and by its continuity, its range in these neighborhoods is the whole unit interval. Thus there are at least two intervals $\{[v_{i-j}^m, v_{i+1}^M]\}_{i=1}^{I-1+j}$, $i = 1, 2$ as defined in the above algorithm. If the two signals satisfying (35) for each interval are such that $s_1 < s_2$, there exists a revealing region in the interval $[\overline{\nu}(s_1), \underline{\nu}(s_2)]$ with $\underline{\nu}(s_2)$, $\overline{\nu}(s_1)$ given by (36) and (37), respectively.

Hence, if $H'(v) > 0$ for any value v such that $H(v) = 1 - \gamma$, then $\overline{\eta} < 1$ whereas $\overline{\eta}$ might be equal to one otherwise. ■

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